Towards quasi-final coalgebraic semantics

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Motivating example

Coalgebraic semantics of Prolog

- Horn clause logic as a programming language in non-deterministic.
- The Prolog machine, however, behaves deterministically, generating in sequence answers to a query by taking advantage of the orderings of clauses and goals.
- A simple model for this machine is a coalgebra of type

$$\psi : S \to \underbrace{S + (A \times S) + \{*\}}_{F(S)} .$$

- What are the behaviours of this coalgebra?

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The Prolog example

Behaviours by finality

$$\psi : S \to S + (A \times S) + \{*\}$$

The final F-coalgebra $\langle X, \xi \rangle$ is formed by the infinite and finite streams over

$$\mathbb{N} \times A$$

ending in 0 or ∞ ($0 \neq \infty$):

$$X = (\mathbb{N} \times A)^* \times \{0, \infty\} + (\mathbb{N} \times A)^\omega$$

$$\xi((0, a).\sigma) = (a, \sigma)$$

$$\xi((n + 1, a).\sigma) = (n, a).\sigma$$

$$\xi(0) = *$$

$$\xi(\infty) = \infty$$

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The Prolog example

Abstracting away the number of internal steps

$$\psi : S \to S + (A \times S) + \{*\}$$

We obtain $\langle Z, \zeta \rangle$ as follows:

$$Z = A^* \times \{0, \infty\} + A^\omega$$

$$\zeta(a.\sigma) = (a, \sigma)$$

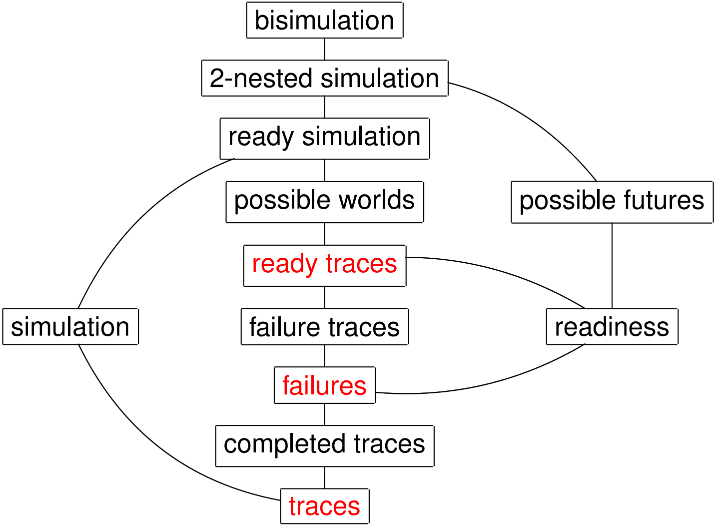
$$\zeta(0) = *$$

$$\zeta(\infty) = \infty$$

$\langle Z, \zeta \rangle$ is a quasi-final F-coalgebra (definition below).

The linear time – branching time spectrum [van Glabbeek 2001]

There is no final transition system. But are there quasi-final ones for these behaviours?



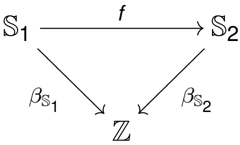
Finality revisited

Using natural transformations

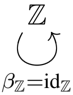
An object $\mathbb{Z}$ in a category **C** is final if there is a natural transformation

$$\beta : I \to \mathbb{Z}$$

from the identity functor I to the constant functor $\mathbb{Z}$



such that $\beta_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$ is the identity morphism:



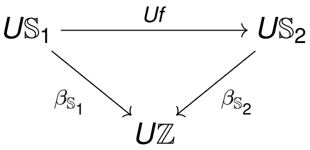
Quasi-final object

In a concrete category **C** with forgetful functor U : **C** $\to$ **Set**

An object $\mathbb{Z}$ in **C** is quasi-final if there is a natural transformation

$$\beta : U \to U\mathbb{Z}$$

from the forgetful functor U to the constant functor $U\mathbb{Z}$



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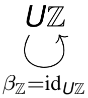

Transition systems

... as coalgebras

- Fixing a set $A \neq \emptyset$ of actions, a (labelled) transition system $\langle S, \rightarrow\rangle$ can be viewed as a coalgebra $\mathbb{S} = \langle S, \psi\rangle$ with $\psi : S \rightarrow \mathcal{P}(S)^A$.
- We shall use the equivalent notations $s \xrightarrow{a} s'$ and $s' \in \psi(s)(a)$ as convenient.
- A morphism $f : \mathbb{S} \rightarrow \mathbb{S}'$ in the coalgebraic setting means that:
  - $s \xrightarrow{a} t$ in $\mathbb{S}$ implies $f(s) \xrightarrow{a} f(t)$ in $\mathbb{S}'$;
  - $f(s) \xrightarrow{a} s'$ in $\mathbb{S}'$ implies $s \xrightarrow{a} t$ in $\mathbb{S}$ and $f(t) = s'$ for some t.

There is no final transition system

A final $\mathbb{Z} = \langle Z, \zeta\rangle$ would have an isomorphism $\zeta : Z \cong \mathcal{P}(Z)^A$ [Lambek], which is impossible for cardinality reasons.

Traces [Hoare 1980]

The transition system of traces is quasi-final

Let $\mathbb{S} = \langle S, \psi \rangle$ be a transition system.

- Write $s \xrightarrow{x} t$ if $s \xrightarrow{a_1} \cdots \xrightarrow{a_n} t$ with $x = a_1 \cdots a_n$.
- The set of traces of s is $Tr_{\mathbb{S}}(s) = \{x \mid \exists\, t, s \xrightarrow{x} t\}$.
- $\varepsilon \in Tr_{\mathbb{S}}(s)$ and $Tr_{\mathbb{S}}(s)$ is prefix-closed.
- A trace language is a nonempty and prefix-closed $L \subseteq A^*$.
- The set $\mathcal{T}$ of trace languages is turned into a transition system $\mathbb{T} = \langle \mathcal{T}, \zeta_{\mathbb{T}} \rangle$ by defining $\zeta_{\mathbb{T}} : \mathcal{T} \to \mathcal{P}(\mathcal{T})^A$ for $a \in L$ by:

$$L \xrightarrow{a} \{x \mid ax \in L\}$$

- For L in $\mathcal{T}$, $Tr_{\mathbb{T}}(L) = L$, that is, $Tr_{\mathbb{F}} = \mathrm{id}_{U\mathbb{T}}$.
- $Tr : U \to U\mathbb{T} = \mathcal{T}$ is a natural transformation, that is, $Tr_{\mathbb{S}}(f(s)) = Tr_{\mathbb{S}}(s)$ for any morphism f.

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Failures (I) [Brookes, Hoare, Roscoe 1984]

Failure-sets

Again $\mathbb{S} = \langle S, \psi \rangle$ is a transition system.

- $(x, X) \in A^* \times \mathcal{P}(A)$ is a failure of s if $s \xrightarrow{x} s'$ and $X \cap I(s') = \emptyset$ for some s', where $I(s) = \{a \in A \mid s \xrightarrow{a}\}$.
- The set of failures of s is written $Fl_{\mathbb{S}}(s)$.
- A failure-set is a set $F \subseteq A^* \times \mathcal{P}(A)$ satisfying the following conditions:
  - $(\varepsilon, \emptyset) \in F$.
  - $(\varepsilon, X) \in F$ implies $\forall a \in X, (a, \emptyset) \notin F$.
  - $(xy, X) \in F$ implies $(x, \emptyset) \in F$.
  - $(x, Y) \in F$ and $X \subseteq Y$ imply $(x, X) \in F$.
  - $(x, X) \in F$ and $(x, X \cup Y) \notin F$ imply $\exists a \in Y, (xa, \emptyset) \in F$.

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Failures (II)

The transition system of failures is quasi-final

- For a failure-set F let $C_F(x) = \{a \in A \mid (xa, \emptyset) \in F\}$.
- $(x, X) \in F$ is a primary failure if $X \subseteq C_F(x)$.
- If $(x, X) \in F$ is a primary failure define $F \xrightarrow{a,X} F'$ by:

$$
\begin{aligned}
F' \;=\; & \{(\varepsilon, Y) \mid Y \subseteq X \cup (A - C_F(a))\} \\
& \cup \\
& \{(bx, Y) \mid (abx, Y) \in F, b \notin X\}.
\end{aligned}
$$

- The set $\mathcal{F}$ of failure-sets gives a transition system $\mathbb{F} = \langle \mathcal{F}, \zeta_{\mathbb{F}} \rangle$ with $\zeta_{\mathbb{F}} : \mathcal{F} \to \mathcal{P}(\mathcal{F})^A$ defined by:

$$
F \xrightarrow{a} F' \quad \text{iff} \quad F \xrightarrow{a,X} F' \quad \text{for some } X.
$$

- For F in $\mathcal{F}$, $Fl_{\mathbb{F}}(F) = F$, that is, $Fl_{\mathbb{F}} = \mathrm{id}_{U\mathbb{F}}$.
- $Fl : U \to U\mathbb{F} = \mathcal{F}$ is a natural transformation, that is, $Fl_{\mathbb{S}}(f(s)) = Fl_{\mathbb{S}}(s)$ for any morphism f.

Some properties of quasi-final objects

Many more remain to be found...

- Let $\langle\mathbb{Z},\beta\rangle$ be a quasi-final object of a concrete category **C**. A function $f : U\mathbb{S}_1 \to U\mathbb{S}_2$ is a β-map if $\beta_{\mathbb{S}_2} \circ f = \beta_{\mathbb{S}_1}$.
  - Every morphism $f : \mathbb{S}_1 \to \mathbb{S}_2$ is a β-map.
  - The objects of **C** with β-maps as morphisms form a category $\mathbf{C} \downarrow \beta$.
  - $\mathbb{Z}$ is a final object of $\mathbf{C} \downarrow \beta$.
- Any object $\mathbb{Z}$ with $U\mathbb{Z} = 1$ is quasi-final.
- If $\langle\mathbb{Z},\beta\rangle$ and $\langle\mathbb{W},\gamma\rangle$ are quasi-final objects and every $\beta_{\mathbb{S}}$ is a γ-map, then $W \xrightarrow{\beta_{\mathbb{W}}} Z \xrightarrow{\gamma_{\mathbb{Z}}} W = \mathrm{id}_{\mathbb{W}}$. So $\mathbb{W}$ is a sub-object of $\mathbb{Z}$ in $\mathbf{C} \downarrow \gamma$.
- In the same conditions, the behavioural equivalences $\overset{\mathbb{Z}}{=}_{\mathbb{S}} = \mathrm{Ker}\ (\beta_{\mathbb{S}})$ and $\overset{\mathbb{W}}{=}_{\mathbb{S}} = \mathrm{Ker}\ (\gamma_{\mathbb{S}})$ satisfy $\overset{\mathbb{Z}}{=}_{\mathbb{S}} \subseteq \overset{\mathbb{W}}{=}_{\mathbb{S}}$.
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Some properties of quasi-final objects

Many more remain to be found...

- Let $\langle \mathbb{Z}, \beta \rangle$ be a quasi-final object of a concrete category **C**. A function $f : U\mathbb{S}_1 \to U\mathbb{S}_2$ is a β-map if $\beta_{\mathbb{S}_2} \circ f = \beta_{\mathbb{S}_1}$.
  - Every morphism $f : \mathbb{S}_1 \to \mathbb{S}_2$ is a β-map.
  - The objects of **C** with β-maps as morphisms form a category $\mathbf{C} \downarrow \beta$.
  - $\mathbb{Z}$ is a final object of $\mathbf{C} \downarrow \beta$.
- Any object $\mathbb{Z}$ with $U\mathbb{Z} = 1$ is quasi-final.
- If $\langle \mathbb{Z}, \beta \rangle$ and $\langle \mathbb{W}, \gamma \rangle$ are quasi-final objects and every $\beta_{\mathbb{S}}$ is a γ-map, then $W \xrightarrow{\beta_{\mathbb{W}}} Z \xrightarrow{\gamma_{\mathbb{Z}}} W = \mathrm{id}_{\mathbb{W}}$. So $\mathbb{W}$ is a sub-object of $\mathbb{Z}$ in $\mathbf{C} \downarrow \gamma$.
- In the same conditions, the behavioural equivalences $\overset{\mathbb{Z}}{=}_{\mathbb{S}} = \mathrm{Ker}\ (\beta_{\mathbb{S}})$ and $\overset{\mathbb{W}}{=}_{\mathbb{S}} = \mathrm{Ker}\ (\gamma_{\mathbb{S}})$ satisfy $\overset{\mathbb{Z}}{=}_{\mathbb{S}} \subseteq \overset{\mathbb{W}}{=}_{\mathbb{S}}$.
- Example: $Tr_{\mathbb{F}}(Fl_{\mathbb{S}}(s)) = Tr_{\mathbb{S}}(s)$, so $\overset{\mathbb{F}}{=}_{\mathbb{S}} \subseteq \overset{\mathbb{T}}{=}_{\mathbb{S}}$.

An approach for defining quasi-final objects

A categorical formulation

Given:

- a concrete category **C**, U;
- an object $\mathbb{Z}$ of **C**.

Find:

- a subcategory **D** having $\mathbb{Z}$ as final object with behaviour $\gamma : \mathrm{Id}_{\mathbf{D}} \to \mathbb{Z}$ in **D**;
- a functor $T : \mathbf{C} \to \mathbf{C}$ with image in **D**
- and a natural transformation $\eta : U \to UT$ such that $\eta_{\mathbb{S}} : U\mathbb{S} \to UT\mathbb{S}$ is a morphism for every $\mathbb{S}$ in **D**.

Then:

$\langle \mathbb{Z}, \beta \rangle$ is a quasi-final object where $\beta_{\mathbb{S}} = U\mathbb{S} \xrightarrow{\eta_{\mathbb{S}}} UT\mathbb{S} \xrightarrow{\gamma_{T\mathbb{S}}} U\mathbb{Z}$.

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Application to previous examples (I)

Prolog example and traces

Prolog example

Take **D** to be the category of the $\langle S, \psi \rangle$ with:

- $\psi : S \to S + (A \times S) + \{*\}$ such that $\psi(s) \in S$ implies $\psi(s) = s$;
- T applies a system $\langle S, \psi \rangle$ to the system $\langle S, \phi \rangle$ where ϕ is equal to ψ except that $\phi(s) = s$ if ψ diverges on s;
- $\eta_{\langle S, \psi \rangle}$ is the identity function.

Traces

Here **D** is the category of deterministic transition systems, T is the familiar (non-empty) powerset construction and $\eta_{\mathbb{S}}(s) = \{s\}$.

Application to previous examples (II)

Failures

Failures

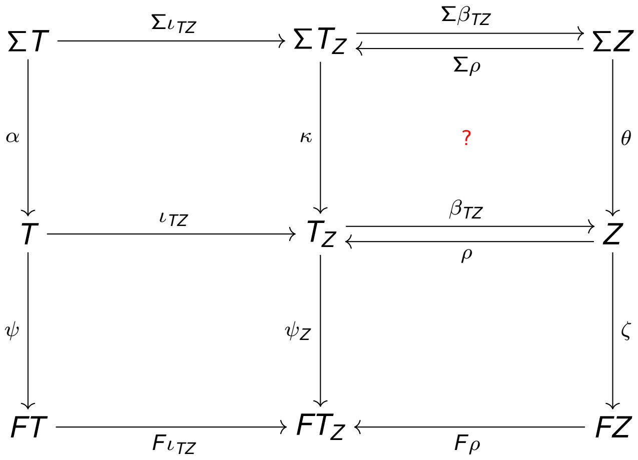
D is the full subcategory of ts's such that:

- $s \xrightarrow{a} t_1$ and $s \xrightarrow{a} t_2$ and $I(t_1) = I(t_2)$ imply $t_1 = t_2$;
- $s \xrightarrow{a} t$ and $I(t) \subseteq J \subseteq C_s(a)$ imply $s \xrightarrow{a} t'$ and $I(t') = J$ for some t';
- $s \xrightarrow{a} s_i \xrightarrow{b} t_i$ $(i = 1, 2)$ imply $s_i \xrightarrow{b} t_{3-i}$ $(i = 1, 2)$.

(T and η omitted.)

Future work: Quasi-final semantics

Following [Rutten Turi 94] [Turi Plotkin 97]



$$\iota_Z \stackrel{?}{=} \beta_{TZ} \circ \iota_{TZ} = \beta_T$$