Calculating from Alloy relational models

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Model driven engineering

- **MEDEA** project — *High Assurance MDE using Alloy*
- **MDE** is a clumsy area of work, full of approaches, acronyms, notations.
- **UML** has taken the lead in *unifying* such notations, but it is too **informal** to be accepted as a reference approach.
- Model-oriented formal methods (**VDM**, **Z**) solve this informality problem at a high-cost: people find it hard to understand models written in maths (cf. maths illiteracy if not mathphobic behaviour).
- **Alloy** [2] offers a good compromise — it is formal in a light-weight manner.

Inspiration

- **BBI** project [3]: **Alloy** re-engineering of a well-tested, very well written non-trivial prototype in **Haskell** of a real-estate trading system similar to the stocks market (65 pages in lhs format) unveiled 4 bugs (2 invariant violations + 2 weak pre-conditions)
- Alloy and Haskell complementary to each other

Real Estate Exchange

Bolsa de Bens Imobiliários

PortoDigital – SEC-11

Joost Visser

Confidential

Draft of August 19, 2007

Alloy

What **Alloy** offers

- A unified approach to **modeling** based on the notion of a **relation — “everything is a relation”** in Alloy.
- A minimal syntax (centered upon relational composition) with an object-oriented flavour which captures much of what otherwise would demand for **UML+OCL**.
- A **pointfree** subset.
- A model-checker for model assertions (counter-examples within scope).

Alloy

What **Alloy does not** offer

- Complete calculus for deduction (proof theory)
- Strong type checking
- Dynamic semantics modeling features

Opportunities

- Enrich the standard Alloy *modus operandi* with relational algebra calculational proofs
- Design an Alloy-centric tool-chain for high assurance model-oriented design

Thus the **MEDEA** project (submitted).

Alloy

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Relational composition

- The swiss army knife of Alloy
- It subsumes function application and "field selection"
- Encourages a navigational (point-free) style based on pattern $x.(R.S)$.
- Example:

$$
\begin{aligned}
Person &= \{(P1),(P2),(P3),(P4)\}\\
parent &= \{(P1,P2),(P1,P3),(P2,P4)\}\\
me &= \{(P1)\}\\
me.parent &= \{(P2),(P3)\}\\
me.parent.parent &= \{(P4)\}\\
Person.parent &= \{(P2),(P3),(P4)\}
\end{aligned}
$$

When "everything is a relation"

- Sets are relations of arity 1, eg. $Person = \{(P1), (P2), (P3), (P4)\}$
- Scalars are relations with size 1, eg. $me = \{(P1)\}$
- Relations are first order, but we have multi-ary relations.
- However, **Alloy** relations are not *n*-ary in the usual sense: instead of thinking of $R \in 2^{A \times B \times C}$ as a set of triples (there is no such thing as *tupling* in Alloy), think of R in terms of *currying*:

$$R \in (B \to C)^A$$

(More about this later.)

Kleene algebra flavour

Basic operators:

.	*composition*
+	*union*
^	*transitive closure*
*	*transitive-reflexive closure*

(There is no recursion is Alloy.) Other relational operators:

~	*converse*
++	*override*
&	*intersection*
-	*difference*
->	*cartesian product*
<:	*domain restriction*
:>	*range restriction*

Relational thinking

- As a rule, thinking in terms of poinfree relations (this includes **functions**, of course) pays the effort: the concepts and the reasoning become simpler.
- This includes **relational data** structuring, which is far more interesting than what can be found in SQL and relational databases.

Example — list processing

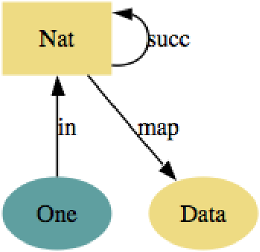
- **Lists** are traditionally viewed as recursive (linear) data structures.
- There are no lists in Alloy — they have to be modeled by **simple** relations (vulg. partial functions) between indices and elements.

Lists as relations in Alloy

```
sig List {
        map : Nat -> lone Data
}

sig Nat {
    succ: one Nat
}

one sig One in Nat {}
```



Multiplicities: `lone` (one or less), `one` (exaclty one)

Relational data structuring

Some correspondences:

list l	**relation** L
sorted	monotonic
noDuplicates	injective
$map\ f\ l$	$f \cdot L$
$zip\ l_1\ l_2$	$\langle L_1, L_2 \rangle$
$[1, \ldots]$	id

where

- id is the identity (equivalence) relation
- the "fork" (also known as "split") combinator is such that $(x, y)\langle L_1, L_2 \rangle z$ means the same as $xL_1z \wedge yL_2z$

Haskell versus Alloy

Pointwise Haskell:

```
findIndices        :: (a -> Bool) -> [a] -> [Int]
findIndices p xs = [ i | (x,i) <- zip xs [0..], p x ]
```

Pointfree (PF):

$$findIndices\ p\ L \quad \triangleq \quad \pi_2 \cdot (\Phi_p \times id) \cdot \langle L, id \rangle \tag{1}$$

where

- π_2 is the right projection of a pair
- $L \times R = \langle L \cdot \pi_1, R \cdot \pi_2 \rangle$
- $\Phi_p \subseteq id$ is the coreflexive relation (partial identity) which models predicate p (or a set)

Haskell versus Alloy

- What about Alloy? It has no pairs, therefore no forks $\langle L, R \rangle \ldots$
- Fortunately there is the relational calculus:

$$
\begin{array}{ll}
 & \pi_2 \cdot (\Phi_p \times id) \cdot \langle L, id \rangle \\
\Leftrightarrow & \quad \{ \ \times\text{-absorption} \ \} \\
 & \pi_2 \cdot \langle \Phi_p \cdot L, id \rangle \\
\Leftrightarrow & \quad \{ \ \times\text{-cancelation} \ \} \\
 & \delta\,(\Phi_p \cdot L)
\end{array}
$$

where $\delta\, R = R^\circ \cdot R \cap id$, for R° the converse of R.

Haskell versus Alloy

Two ways of writing $\delta\,(\Phi_p \cdot L)$ in Alloy, one pointwise

```
fun findIndices[s:set Data,l:List]: set Nat {
      {i: Nat | some x:s | x in i.(l.map)}
}
```

and the other pointfree,

```
fun findIndices[s:set Data,l:List]: set Nat {
      dom[l.map :> s]
}
```

the latter very close to what we've calculated.

Beyond model-checking: proofs by calculation

Suppose the following property

$$(\mathit{findIndices}\ p) \cdot r^{\star} \quad = \quad \mathit{findIndices}\ (p \cdot r) \tag{2}$$

is asserted in Alloy:

```
assert FT {
    all l,l':List, p: set Data, r: Data -> one Data |
        l'.map = l.map.r =>
            findIndices[p,l'] = findIndices[r.p,l]
}
```

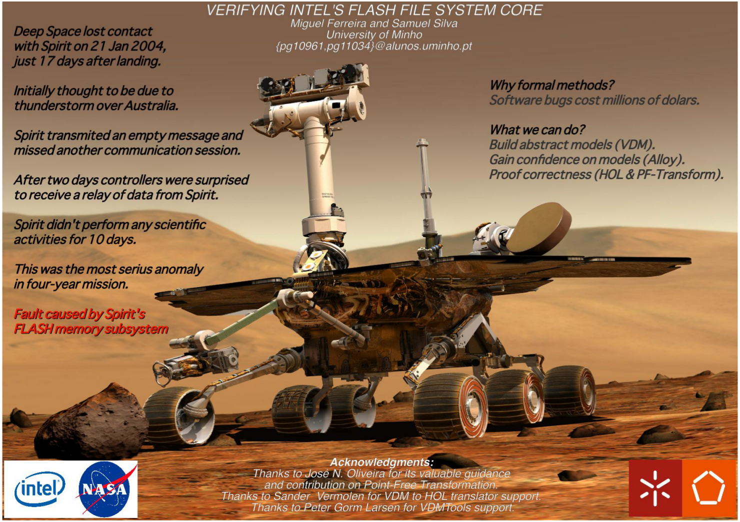
and that the model checker does not yield any counter-examples. How can we be sure of its validity?

- Free theorems — the given assertion is a corollary of the free theorem of *findIndices*, thus there is nothing to prove (model checking could be avoided!)
- Wishing to prove the assertion anyway, one calculates:

Trivial proof

$$
\begin{array}{ll}
 & (\mathit{findIndices}\ p) \cdot r^{\star} = \mathit{findIndices}\ (p \cdot r) \\
\Leftrightarrow & \quad \{ \text{ list to relation transform } \} \\
 & \delta\,(\Phi_p \cdot (r \cdot L)) = \delta\,(\Phi_{p \cdot r} \cdot L) \\
\Leftrightarrow & \quad \{ \text{ property } \Phi_{f \cdot g} = \delta\,(\Phi_f \cdot g) \ \} \\
 & \delta\,(\Phi_p \cdot (r \cdot L)) = \delta\,(\delta\,(\Phi_p \cdot r) \cdot L) \\
\Leftrightarrow & \quad \{ \text{ domain of composition } \} \\
 & \delta\,(\Phi_p \cdot (r \cdot L)) = \delta\,((\Phi_p \cdot r) \cdot L) \\
\Leftrightarrow & \quad \{ \text{ associativity } \} \\
 & \textsc{True}
\end{array}
$$

Realistic example — Verified FSystem (VFS)



VFS in Alloy (simplified)

The system:

```
sig System {
  fileStore: Path -> lone File,
  table: FileHandle -> lone OpenFileInfo
}
```

Paths:

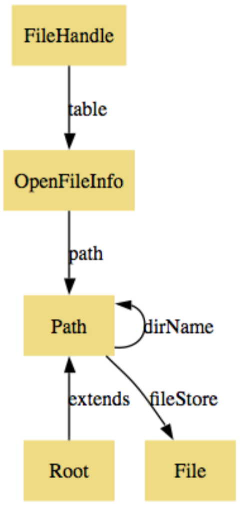
```
sig Path {
  dirName: one Path
}
```

The root is a path:

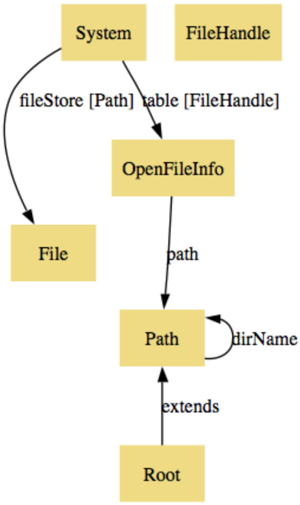
```
one sig Root extends Path {
}
```

Alloy diagrams for FSystem

Simplified:



Complete:



Binary relation semantics

Meaning of signatures:

```
sig Path {
  dirName: one Path
}
```

declares function $Path \xrightarrow{dirName} Path$.

```
sig System {
  fileStore: Path -> lone File,
}
```

declares simple relation $System \times Path \xrightarrow{fileStore} File$.
(**NB**: a relation S is **simple**, or *functional*, wherever its **image** $S \cdot S^\circ$ is coreflexive. Using harpoon arrows $\rightharpoonup$ for these.)

Binary relation semantics

- Since

$$(A \times B) \rightharpoonup C \quad \cong \quad (B \rightharpoonup C)^A$$

$\mathit{fileStore}$ can be alternatively regarded as a function in $(\mathit{Path} \rightharpoonup \mathit{File})^{\mathit{System}}$, that is, for $s : \mathit{System}$,

$$\mathit{Path} \xrightharpoonup{(\mathit{fileStore}\ \ s)} \mathit{File}$$

- Thus the "navigation-styled" notation of Alloy: $p.(s.\mathit{fileStore})$ means the file accessible from path p in file system s.
- Similarly, line `table: FileHandle -> lone OpenFileInfo` in the model declares

$$\mathit{FileHandle} \xrightharpoonup{(\mathit{table}\ \ s)} \mathit{OpenFileInfo}$$

Multiplicities in Alloy + taxonomy

A lone -> B	A -> some B	A -> lone B	A some -> B
injective	entire	simple	surjective

A lone -> some B	A -> one B	A some -> lone B
representation	function	abstraction

A lone -> one B	A some -> one B
injection	surjection

A one -> one B
bijection

(courtesy of Alcino Cunha, the Alloy expert at Minho)

Terminology reminder

Topmost criteria:

binary relation

injective entire simple surjective

Definitions:

	Reflexive	*Coreflexive*
$\ker R$	entire R	injective R
$\mathrm{img}\, R$	surjective R	simple R

$$\ker R = R^{\circ} \cdot R$$
$$\mathrm{img}\, R = R \cdot R^{\circ}$$

From Alloy to relational diagrams

$$
\begin{tikzcd}
\color{red}{OpenFileDescriptor} \arrow[d, "path"'] & & \color{red}{FileHandle} \arrow[ll, "\color{red}{table\ \ s}"'] \\
\color{red}{Path} \arrow[rr, "\color{red}{fileStore\ \ s}"'] & & \color{red}{File} \\
Path \arrow[u, "dirName"] & &
\end{tikzcd}
$$

where

- *table s*, *fileStore s* are simple relations
- the other arrows depict functions

(diagram in the **Rel** allegory to be completed)

Model constraints

Referential integrity:

Non-existing files cannot be opened:

```
pred ri[s: System]{
    all h: FileHandle, o: h.(s.table) |
                      some (o.path).(s.fileStore)
}
```

Paths closure:

Mother directories exist and are indeed directories:

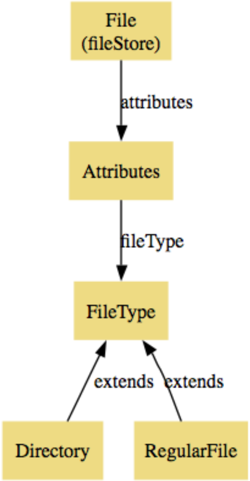
```
pred pc[s: System]{
  all p: Path |
        some p.(s.fileStore) =>
        (some d: (p.dirName).(s.fileStore) |
                                 d.fileType=Directory)
}
```

2nd part of Alloy FSystem model

```
sig File {
    attributes: one Attributes
}

sig Attributes{
    fileType: one FileType
}

abstract sig FileType {}
one sig RegularFile extends FileTy
one sig Directory extends FileType
```



Updated binary relational diagram

$$
\begin{tikzcd}
OpenFileDescriptor \arrow[d, "path"'] & & FileHandle \arrow[ll, "table\ \ s"'] & \\
Path \arrow[rr, "fileStore\ \ s"'] & & File \arrow[r, "attributes"] & Attributes \arrow[d, "fileType"] \\
Path \arrow[u, "dirName"] & & & FileType
\end{tikzcd}
$$

where

- *table s*, *fileStore s* are simple relations
- all the other arrows depict functions

Constraints: still missing

Updating diagram with constraints

Complete diagram, where M abbreviates *table s*, N abbreviates *fileStore s* and $\underline{k}$ is the "everywhere-k" function:

$$
\begin{tikzcd}
OpenFileDescriptor \arrow[d, "path"'] \arrow[dr, phantom, "\subseteq"] & FileHandle \arrow[l, "M"'] \arrow[d, "\top"] & \\
Path \arrow[r, shift right, "N"'] & File \arrow[l, shift right, "N^{\circ}"'] \arrow[r, "attributes"] \arrow[d, phantom, "\rotatebox{90}{$\subseteq$}"] & Attributes \arrow[d, "fileType"] \\
Path \arrow[u, "dirName"] \arrow[r, "N"'] & File \arrow[r, "\underline{Directory}"'] & FileType
\end{tikzcd}
$$

Constraints:

- Top rectangle is the PF-transform of *ri* (referential integrity)
- Bottom rectangle is the PF-transform of *pc* (path closure)

PF-constraints in symbols

Referential integrity:

$$ri(M, N) \quad \triangleq \quad path \cdot M \subseteq N^{\circ} \cdot \top \tag{3}$$

which is equivalent to

$$ri(M, N) \quad \triangleq \quad \rho\,(path \cdot M) \subseteq \delta\, N$$

where $\rho\, R = \delta\, R^{\circ}$. PF version (3) also easy to encode in Alloy

```
pred riPF[s: System]{
    s.table.path in (FileHandle->File).~(s.fileStore)
}
```

thanks to its emphasis on **composition**.

PF-constraints in symbols

Referential integrity:

$$ri(M, N) \triangleq path \cdot M \subseteq N^{\circ} \cdot \top \quad (3)$$

which is equivalent to

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```
pred riPF[s: System]{
    s.table.path in (FileHandle->File).~(s.fileStore)
}
```

thanks to its emphasis on **composition**.

PF-constraints in symbols

Paths closure:

$$pc\ N \triangleq \underline{Directory} \cdot N \subseteq fileType \cdot attributes \cdot N \cdot dirName \qquad (4)$$

recall diagram:

$$\begin{tikzcd}
\color{red}{Path} \arrow[r, "\color{red}{N}"'] & \color{red}{File} \arrow[r, "attributes"] & Attributes \arrow[d, "fileType"] \\
Path \arrow[u, "dirName"] \arrow[r, "N"'] & File \arrow[u, phantom, "\cup\!|"] \arrow[r, "\underline{Directory}"'] & FileType
\end{tikzcd}$$

Again thanks to emphasis on **composition**, this is easily encoded in PF-Alloy:

```
pred pcPF[s: System]{
   s.fileStore.(File->Directory) in
      dirName.(s.fileStore).attributes.fileType
}
```

PF-constraints in symbols

Paths closure:

$$pc\ N \triangleq \underline{Directory} \cdot N \subseteq fileType \cdot attributes \cdot N \cdot dirName \qquad (4)$$

recall diagram:

$$\begin{tikzcd}
Path \arrow[r, "N"'] & File \arrow[r, "attributes"] & Attributes \arrow[d, "fileType"] \\
Path \arrow[u, "dirName"] \arrow[r, "N"'] & File \arrow[u, phantom, "\rotatebox{90}{$\subseteq$}"] \arrow[r, "\underline{Directory}"'] & FileType
\end{tikzcd}$$

Again thanks to emphasis on **composition**, this is easily encoded in PF-Alloy:

```
pred pcPF[s: System]{
   s.fileStore.(File->Directory) in
       dirName.(s.fileStore).attributes.fileType
}
```

PF-ESC by calculation

- Models with constraints put the burden on the designer to ensure that operations **type-check** (read this in **extended**-mode), that is, constraints are preserved across the models operations.
- Typical approach in MDE: **model-checking**
- Automatic **theorem proving** also considered in safety-critical systems
- However: convoluted pointwise formulæ often lead to failure.

How about doing these as "pen & paper" exercises?

- **PF-formulæ** are manageable, this is the difference.

Example of PF-ESC by calculation

Consider the operation which removes file system objects, as modeled in Alloy:

```
pred delete[s',s: System, sp: set Path]{
        s'.table = s.table
        s'.fileStore = (univ-sp) <: s.fileStore
}
```

that is,

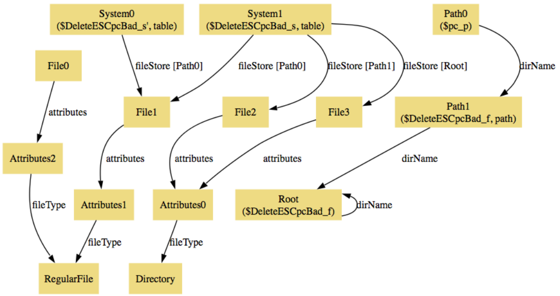
$$delete\ S\ (M, N) \quad \triangleq \quad (M, N \cdot \Phi_{(\notin S)}) \tag{5}$$

where $\Phi_{(\notin S)}$ is the coreflexive associated to the complement of S.

Intuitive steps

Intuitively, *delete* will put the

- *ri* constraint at risk once we decide to delete file system objects which are open
- *pc* constraint at risk once we decide to delete directories with children.



(Model-checking in **Alloy** will easily spot these flaws, as checked above by a counter-example for the latter situation.)

Intuitive steps

We have to guess a **pre-conditions** for *delete*. However,

- How can we be sure that such (guessed) pre-condition is *good enough*?
- The best way is to calculate the weakest pre-condition for each constraint to be maintained.
- In doing this, mind the following properties of relational algebra:

$$h \cdot R \subseteq S \quad \Leftrightarrow \quad R \subseteq h^{\circ} \cdot S \tag{6}$$

$$R \cdot \Phi \quad = \quad R \cap \top \cdot \Phi \tag{7}$$

$$f \cdot R \subseteq \top \cdot S \quad \Leftrightarrow \quad R \subseteq \top \cdot S \tag{8}$$

For improved readability, we introduce abbreviations $ft := fileType \cdot attributes$ and $d := \underline{Directory}$, and **calculate**:

Calculational steps

$$
\begin{array}{ll}
 & pc(delete\ S\ (M, N)) \\
\Leftrightarrow & \quad \{\ (5) \text{ and } (4)\ \} \\
 & d \cdot (N \cdot \Phi_{(\notin S)}) \subseteq ft \cdot (N \cdot \Phi_{(\notin S)}) \cdot dirName \\
\Leftrightarrow & \quad \{\ \text{shunting } (6)\ \} \\
 & d \cdot N \cdot \Phi_{(\notin S)} \cdot dirName^{\circ} \subseteq ft \cdot N \cdot \Phi_{(\notin S)} \\
\Leftrightarrow & \quad \{\ (7)\ \} \\
 & d \cdot N \cdot \Phi_{(\notin S)} \cdot dirName^{\circ} \subseteq ft \cdot N \cap \top \cdot \Phi_{(\notin S)} \\
\Leftrightarrow & \quad \{\ \cap\text{-universal} ; \text{shunting}\ \}
\end{array}
$$

Ensuring paths closure

$$\left\{ \begin{array}{l} d \cdot N \cdot \Phi_{(\notin S)} \subseteq ft \cdot N \cdot dirName \\ d \cdot N \cdot \Phi_{(\notin S)} \subseteq \top \cdot \Phi_{(\notin S)} \cdot dirName \end{array} \right.$$

$$\Leftrightarrow \qquad \{ \ \top \text{ absorbs } d \ (8) \ \}$$

$$\left\{ \begin{array}{l} \underbrace{d \cdot N \cdot \Phi_{(\notin S)} \subseteq ft \cdot N \cdot dirName}_{\text{weaker than } pc(N)} \\ \underbrace{N \cdot \Phi_{(\notin S)} \subseteq \top \cdot \Phi_{(\notin S)} \cdot dirName}_{wp} \end{array} \right.$$

Back to points, *wp* is:

$$\langle \forall\ q \ :\ q \in dom\ N \wedge q \notin S :\ dirName\ q \notin S \rangle$$

$$\Leftrightarrow \qquad \{ \ \text{predicate logic} \ \}$$

$$\langle \forall\ q \ :\ q \in dom\ N \wedge (dirName\ q) \in S :\ q \in S \rangle$$

Ensuring paths closure

In words:

> *if parent directory of existing path q is marked for deletion than so must be q.*

Translating calculated weakest precondition back to Alloy:

```
pred pre_delete[s: System, sp: set Path]{
     all q: Path |
        some q.(s.fileStore) &&
             q.dirName in sp => q in sp
}
```

Back to the diagram

PF-encoding of model constraints in terms of relational composition has at least the following advantages:

- it makes **calculations** easier (rich algebra of $R \cdot S$)
- it makes it possible to **draw** constraints as rectangles in diagrams, recall

$$
\begin{tikzcd}
OpenFileDescriptor \arrow[d, "path"'] & & FileHandle \arrow[ll, "M"'] \arrow[d, "\top"] \arrow[dll, phantom, "\subseteq"] & \\
Path \arrow[rr, shift right, "N"'] & & File \arrow[ll, shift right, "N^\circ"'] \arrow[r, "attributes"] \arrow[d, phantom, "\cup\!|"] & Attributes \arrow[d, "fileType"] \\
Path \arrow[u, "dirName"] \arrow[rr, "N"'] & & File \arrow[r, "\underline{Directory}"'] & FileType
\end{tikzcd}
$$

- it enables the "navigation-styled" notation of Alloy

Constraint bestiary

- Experience in formal modeling tells that designs are **repetitive** in the sense that they instantiate (**generic**) constraints whose ubiquitous nature calls for classification
- Such "**constraint patterns**" are rectangles, thus easy to draw and recall
- In the next slides we browse a little "constraint bestiary" capturing some typical samples.

Constraints are Rectangles

- All of shape

$$R \cdot I \quad \subseteq \quad O \cdot R$$

- Example: **referential integrity** in general, where N is the *offer* and M is the *demand* :

$$\rho\,(\in_{\mathsf{F}} \cdot M) \subseteq \delta\, N \quad \Leftrightarrow \quad \begin{tikzcd} \mathsf{F}\,B \arrow[d, "\in_{\mathsf{F}}"'] \arrow[dr, phantom, "\subseteq"] & A \arrow[l, "M"'] \arrow[d, "\top"] \\ B \arrow[r, shift right, "N"'] & C \arrow[l, shift right, "N^{\circ}"'] \end{tikzcd}$$

$$\Leftrightarrow \quad \in_{\mathsf{F}} \cdot M \subseteq N^{\circ} \cdot \top$$

M, N simple. $\in_{\mathsf{F}}$ is a membership relation.

Constraints are Rectangles

- **Example:** M, N domain-disjoint

$$M \cdot N^{\circ} \quad \subseteq \quad \bot$$

- **Example:** simple M, N domain-coherent

$$M \cdot N^{\circ} \quad \subseteq \quad id$$

- **Example:** M domain-closed by R:

$$M \cdot R^{\circ} \subseteq \top \cdot M$$

(path-closure constraint instance of this)

- **Example:** range of R in Φ

$$R \subseteq \Phi \cdot R$$

Experience and Current work

- Defining a simple pointfree **binary** relational semantics for **Alloy** [1]
- Studying the translation to/from Haskell and, in particular, how to port counterexamples to QuickCheck.
- Designing an Alloy-centric **tool-chain** including a (pointfree) extended static checker, translators to Haskell, UML and SQL.

Closing

Why the **UML+OCL**? Why **ERDs**?

- What one draws in UML and ERDs can be captured by binary relational diagrams — not only the class/entity attributive structure + relationships but also the constraints which one normally can't depict at all
- Drawing a constraint as a rectangle means it's well understood, and that calculations will be easier to carry out (run away from logical $\wedge$ if you can!)
- Rectangles nicely encoded in plain PF-Alloy or hybrid navigation-styled Alloy

As Alan Perlis once wrote down:

"Simplicity does not precede complexity, but follows it."

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