Cálculo de Programas

2.º Ano de LCC (Universidade do Minho)
Ano Lectivo de 2024/25

1º Teste — 22 de Março de 2025 , 09h00–11h00
Salas E2-0.01 + E2-0.03

PROVA PRESENCIAL INDIVIDUAL SEM CONSULTA (2h)

Importante — *Ler antes de iniciar a prova:*

- *Esta prova consta de **8** questões que valem, cada uma, 2.5 valores. O tempo médio estimado para resolução de cada questão é de 15 min.*
- *Recomenda-se que os alunos leiam a prova antes de decidirem por que ordem querem responder às questões que são colocadas.*

Questão 1 Deduza o tipo mais geral da função

$$\mathsf{fun} = \langle \pi_1, id \rangle \cdot \langle \pi_2, id \rangle \tag{E1}$$

e infira a respectiva propriedade *grátis* (*natural*) através de um diagrama.

RESOLUÇÃO: Embora não seja necessário, pode fazer-se, por fusão, a seguinte transformação na função

$$\mathsf{fun} = \langle \pi_2, \langle \pi_2, id \rangle \rangle$$

obtendo-se facilmente o tipo:

$$\mathsf{fun} :: A \times B \to B \times (B \times (A \times B))$$

(Justificar.) Usando o diagrama habitual (há que desenhá-lo) obter-se-á a propriedade grátis:

$$(g \times (g \times (f \times g))) \cdot \mathsf{fun} = \mathsf{fun} \cdot (f \times g)$$

□

Questão 2 Mostre que a expressão

$$\langle [f, h] \cdot (\pi_1 + \pi_1), [g, k] \cdot (\pi_2 + \pi_2) \rangle$$

simplifica em:

$$[f \times g, h \times k].$$

RESOLUÇÃO: Tem-se (justificar os passos com reticências):

$$
\begin{array}{cl}
 & \langle [f\,,h]\cdot(\pi_1+\pi_1), [g\,,k]\cdot(\pi_2+\pi_2)\rangle \\
= & \qquad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
 & \langle [f\cdot\pi_1\,, h\cdot\pi_1], [g\cdot\pi_2\,, k\cdot\pi_2]\rangle \\
= & \qquad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
 & [\langle f\cdot\pi_1, g\cdot\pi_2\rangle\,, \langle h\cdot\pi_1, k\cdot\pi_2\rangle] \\
= & \qquad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
 & [f\times g\,, h\times k]
\end{array}
$$

$\square$

Questão 3 Recorde o isomorfismo

$$
\begin{tikzcd}
\mathit{Maybe}\ B \arrow[r, bend left, "\mathsf{out}=\mathsf{in}^{\circ}"] \arrow[r, phantom, "\cong"] & 1+B \arrow[l, bend left, "\mathsf{in}=[\underline{\mathsf{Nothing}}\,{,}\mathsf{Just}]"]
\end{tikzcd}
$$

e considere a função:

$$
\begin{array}{l}
\mathit{maybeToList} :: \mathit{Maybe}\ a \to [\,a\,] \\
\mathit{maybeToList} = [\mathsf{nil}\,,\mathsf{singl}]\cdot\mathsf{out}
\end{array}
$$

Derive a versão *pointwise* de $\mathit{maybeToList}$ por forma a não recorrer ao combinador de alternativa (vulg. 'either') nem a funções constantes. Recorda-se que $\mathsf{nil}\ \_ = [\,]$ e $\mathsf{singl}\ a = [\,a\,]$.

RESOLUÇÃO: Tem-se (justificar os passos com reticências):

$$
\begin{array}{cl}
 & \mathit{maybeToList} = [\mathsf{nil}\,,\mathsf{singl}]\cdot\mathsf{out} \\
\equiv & \qquad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
 & \mathit{maybeToList}\cdot[\underline{\mathsf{Nothing}}\,,\mathsf{Just}] = [\mathsf{nil}\,,\mathsf{singl}] \\
\equiv & \qquad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
 & \left\{\begin{array}{l} \mathit{maybeToList}\cdot\underline{\mathsf{Nothing}} = \mathsf{nil} \\ \mathit{maybeToList}\cdot\mathsf{Just} = \mathsf{singl} \end{array}\right. \\
\equiv & \qquad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
 & \left\{\begin{array}{l} \mathit{maybeToList}\ \mathsf{Nothing} = [\,] \\ \mathit{maybeToList}\ (\mathsf{Just}\ a) = [\,a\,] \end{array}\right.
\end{array}
$$

$\square$

Questão 4 Seja dado um predicado p e uma função f tal que:

$$p\cdot f = p \tag{E2}$$

Mostre que a seguinte composição de condicionais de McCarthy envolvendo p e f

$$(p\to \mathit{id}\ ,\ f)\cdot(p\to f\ ,\ \mathit{id}) \tag{E3}$$

se pode reduzir a f sabendo-se que, entre outras leis que conhece, se tem:

$$p \to (p \to a\ ,\ b)\ ,\ (p \to c\ ,\ d) \quad = \quad p \to a\ ,\ d \tag{E4}$$

$$p \to a\ ,\ a \quad = \quad a \tag{E5}$$

RESOLUÇÃO:

$$
\begin{array}{cl}
& (p \to id\ ,\ f) \cdot (p \to f\ ,\ id) \\
= & \quad \{\ 1^{\text{a}} \text{ lei de fusão do condicional ; nat-id (1) }\} \\
& p \to ((p \to id\ ,\ f) \cdot f)\ ,\ (p \to id\ ,\ f) \\
= & \quad \{\ 2^{\text{a}} \text{ lei de fusão do condicional ; nat-id (1) }\} \\
& p \to (p \cdot f \to f\ ,\ f \cdot f)\ ,\ (p \to id\ ,\ f) \\
= & \quad \{\ p \cdot f = p \text{ (E2) }\} \\
& p \to (p \to f\ ,\ f \cdot f)\ ,\ (p \to id\ ,\ f) \\
= & \quad \{\ \text{lei (E4) acima }\} \\
& p \to f\ ,\ f \\
= & \quad \{\ \text{lei (E5) acima }\} \\
& f
\end{array}
$$

□

Questão 5 Nas aulas foi demonstrada a propriedade:

$$\overline{f \cdot (g \times h)} \;=\; \overline{\mathsf{ap} \cdot (id \times h)} \cdot \overline{f} \cdot g \tag{E6}$$

Acrescente variáveis a ambos os lados de (E6) e confirme que obtém o mesmo resultado nos dois lados da igualdade.

RESOLUÇÃO: Tem-se (justificar os passos com reticências):

$$
\begin{array}{cl}
& \overline{f \cdot (g \times h)}\ a = (\overline{\mathsf{ap} \cdot (id \times h)} \cdot \overline{f} \cdot g)\ a \\
\equiv & \quad \{\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \} \\
& \overline{f \cdot (g \times h)}\ a = \overline{\mathsf{ap} \cdot (id \times h)}\ (\overline{f}\ (g\ a)) \\
\equiv & \quad \{\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \} \\
& \overline{f \cdot (g \times h)}\ a\ b = \overline{\mathsf{ap} \cdot (id \times h)}\ (\overline{f}\ (g\ a))\ b \\
\equiv & \quad \{\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \} \\
& (f \cdot (g \times h))\ (a, b) = (\mathsf{ap} \cdot (id \times h))\ (\overline{f}\ (g\ a), b) \\
\equiv & \quad \{\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \} \\
& f\ (g\ a, h\ b) = \mathsf{ap}\ (\overline{f}\ (g\ a), h\ b) \\
\equiv & \quad \{\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \} \\
& f\ (g\ a, h\ b) = \overline{f}\ (g\ a)\ (h\ b) \\
\equiv & \quad \{\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \} \\
& f\ (g\ a, h\ b) = f\ (g\ a, h\ b)
\end{array}
$$

$\square$

Questão 6 Considere a seguinte proposta de um tipo de dados em Haskell que representa sequências não vazias de *bits*, ie. Booleanos ($\mathbb{B}$):

$$\textbf{data}\ S = \mathit{Sing}\ \mathbb{B} \mid \mathit{Add}\ \mathbb{B}\ S \tag{E7}$$

Defina o isomorfismo in tal como aparece no diagrama,

$$\begin{tikzcd} S \arrow[rr, bend left, "\mathsf{out}=\mathsf{in}^{\circ}"] \arrow[rr, phantom, "\cong"] & & \mathbb{B}+\mathbb{B}\times S \arrow[ll, bend left, "\mathsf{in}"] \end{tikzcd}$$

e calcule out a partir dessa definição.

RESOLUÇÃO: Tem-se $\mathsf{in} = [\mathit{Sing}\ , \widehat{\mathit{Add}}]$, logo (justificar passos):

$$\begin{aligned}
& \mathsf{out} \cdot [\mathit{Sing}\ , \widehat{\mathit{Add}}] = id \\
\equiv \quad & \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& [\mathsf{out} \cdot \mathit{Sing}\ , \mathsf{out} \cdot \widehat{\mathit{Add}}] = id \\
\equiv \quad & \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& \begin{cases} \mathsf{out} \cdot \mathit{Sing} = i_1 \\ \mathsf{out} \cdot \widehat{\mathit{Add}} = i_2 \end{cases} \\
\equiv \quad & \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& \begin{cases} \mathsf{out}\ (\mathit{Sing}\ b) = i_1\ b \\ \mathsf{out}\ (\widehat{\mathit{Add}}\ (b, s)) = i_2\ (b, s) \end{cases} \\
\equiv \quad & \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& \begin{cases} \mathsf{out}\ (\mathit{Sing}\ b) = i_1\ b \\ \mathsf{out}\ (\mathit{Add}\ b\ s) = i_2\ (b, s) \end{cases}
\end{aligned}$$

$\square$

Questão 7 Considere a igualdade funcional

$$f = [\underline{0}, \mathsf{add}] \cdot (id + id \times f) \cdot (id + \langle g, id \rangle) \cdot \mathsf{out}_{\mathbb{N}_0} \tag{E8}$$

onde $\mathsf{out}_{\mathbb{N}_0}$ é a inversa de $\mathsf{in}_{\mathbb{N}_0} = [\underline{0}, \mathsf{succ}]$, $\mathsf{add} = \widehat{(+)}$ e $g\ n = 2 * n + 1$. Mostre, usando as leis do Cálculo de Programas, que (E8) é equivalente a:

$$\begin{aligned} & f\ 0 = 0 \\ & f\ (n+1) = 2 * n + 1 + f\ n \end{aligned}$$

Que faz a função f? Responda informalmente.

RESOLUÇÃO: Tem-se:

$$
\begin{array}{ll}
& f = [\underline{0}, \mathsf{add}] \cdot (id + id \times f) \cdot (id + \langle g, id \rangle) \cdot \mathsf{out}_{\mathbb{N}_0} \\
\equiv & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& f \cdot \mathsf{in}_{\mathbb{N}_0} = [\underline{0}, \mathsf{add} \cdot (id \times f)] \cdot (id + \langle g, id \rangle) \\
\equiv & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& f \cdot [\mathsf{zero}, \mathsf{succ}] = [\underline{0}, \mathsf{add} \cdot (id \times f) \cdot \langle g, id \rangle] \\
\equiv & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& [f \cdot \mathsf{zero}, f \cdot \mathsf{succ}] = [\underline{0}, \mathsf{add} \cdot \langle g, f \rangle] \\
\equiv & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& \left\{ \begin{array}{l} f\ \underline{0} = \underline{0} \\ f \cdot \mathsf{succ} = \mathsf{add} \cdot \langle g, f \rangle \end{array} \right. \\
\equiv & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& \left\{ \begin{array}{l} f\ \underline{0} = \underline{0} \\ f\ (n+1) = \mathsf{add}\ (\langle g, f \rangle\ n) \end{array} \right. \\
\equiv & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& \left\{ \begin{array}{l} f\ \underline{0} = \underline{0} \\ f\ (n+1) = (g\ n) + (f\ n) \end{array} \right. \\
\equiv & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& \left\{ \begin{array}{l} f\ 0 = 0 \\ f\ (n+1) = (2\ n + 1) + f\ n \end{array} \right.
\end{array}
$$

A função calcula os quadrados dos seus *inputs* (Porquê?). □

Questão 8 Mostre que o resultado de aplicar o ciclo-for

$$\mathsf{for}\ (a+)\ b$$

a n é $b + a \times n$, isto é

$$\mathsf{for}\ (a+)\ b \quad = \quad k \tag{E9}$$

onde $k\ n = b + a \times n$. **Sugestão**: recorde a definição

$$\mathsf{for}\ b\ i = (\!|\, [\underline{i}, b] \,|\!) \tag{E10}$$

e use a lei universal-cata. (Apoie a sua resolução num diagrama.)

RESOLUÇÃO: Tem-se (justificar os passos que são dados):

$$
\begin{array}{ll}
& \mathsf{for}\ (a+)\ b = k \\
\equiv & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& k = (\!|\, [\underline{b}, (a+)] \,|\!) \\
\equiv & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& k \cdot \mathsf{in}_{\mathbb{N}_0} = [\underline{b}, (a+)] \cdot (id + k)
\end{array}
$$

$$
\begin{array}{ll}
\equiv & \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ \} \\
& k \cdot [\underline{0}, \mathsf{succ}] = [\underline{b}, (a+) \cdot k] \\
\equiv & \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ \} \\
& \left\{ \begin{array}{l} k \cdot \underline{0} = \underline{b} \\ k \cdot \mathsf{succ} = (a+) \cdot k \end{array} \right. \\
\equiv & \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ \} \\
& \left\{ \begin{array}{l} b + a \times 0 = b \\ b + a \times (n+1) = a + (b + a \times n) \end{array} \right. \\
\equiv & \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ \} \\
& \left\{ \begin{array}{l} b = b \\ b + a\ n + a = a + b + a\ n \end{array} \right.
\end{array}
$$

$\square$
