Cálculo de Programas

3.º Ano de LEI+MiEI (Universidade do Minho)
Ano Lectivo de 2024/25

2º Teste — 18 de Dezembro de 2024, 15h30–17h30
Salas E1-0.08 + E1-1.05 + E2-0.07

PROVA PRESENCIAL INDIVIDUAL SEM CONSULTA (2h)

Importante — *Ler antes de iniciar a prova:*

- *Esta prova consta de **8** questões que valem, cada uma, 2.5 valores. O tempo médio estimado para resolução de cada questão é de 15 min.*
- *Recomenda-se que os alunos leiam a prova antes de decidirem por que ordem querem responder às questões que são colocadas.*

Questão 1 Recordando das aulas a definição

$$\mathsf{for}\ b\ i = (\!|\, [\underline{i}\,, b] \,|\!) \tag{E1}$$

calcule, por fusão cata, a *side condition* da propriedade:

$$h \cdot (\mathsf{for}\ g\ k) = \mathsf{for}\ j\ (h\ k) \quad \Leftarrow \quad \cdots\cdots$$

RESOLUÇÃO: Tem-se (justificar os passos que são dados):

$$\begin{array}{ll} & h \cdot \mathsf{for}\ g\ k = \mathsf{for}\ j\ (h\ k) \\ \equiv & \quad \{ \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ \} \\ & h \cdot (\!|\, [\underline{k}\,, g] \,|\!) = (\!|\, [\underline{h\ k}\,, j] \,|\!) \\ \Leftarrow & \quad \{ \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ \} \\ & h \cdot [\underline{k}\,, g] = [\underline{h\ k}\,, j] \cdot (id + h) \\ \equiv & \quad \{ \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ \} \\ & \left\{ \begin{array}{l} \mathrm{TRUE} \\ h \cdot g = j \cdot h \end{array} \right. \end{array}$$

Logo, a propriedade completa é:

$$h \cdot (\mathsf{for}\ g\ k) = \mathsf{for}\ j\ (h\ k) \quad \Leftarrow \quad h \cdot g = j \cdot h$$

□

Questão 2 Seja dada a função, em Haskell:

$$
\begin{array}{l}
look :: Eq\ k \Rightarrow k \rightarrow [(k,a)] \rightarrow Maybe\ a \\
look\ k\ [\,] = \mathsf{Nothing} \\
look\ k\ ((a,b):r) \\
\quad \mid a \equiv k = \mathsf{Just}\ b \\
\quad \mid \mathsf{otherwise} = look\ k\ r
\end{array}
$$

Defina $look\ k$ como um catamorfismo de listas, isto é, encontre g tal que $look\ k = (\!|\, g\ k \,|\!)$. Apoie a sua resposta num diagrama explicativo. **NB**: recordar que, em listas, $\mathsf{B}\ (f,g) = id + f \times g$ e $\mathsf{in} = [\mathsf{nil}, \mathsf{cons}]$, para $\mathsf{nil}\ \_ = [\,]$ e $\mathsf{cons}\ (a,b) = a:b$.

RESOLUÇÃO: Comecemos por esboçar o diagrama:

$$
\begin{tikzcd}
(K \times A)^* \arrow[d, "look\ k = (\!|\, g\ k \,|\!)"'] \arrow[r, bend left, "\mathsf{out} = \mathsf{in}^\circ"] \arrow[r, phantom, "\cong"] & 1 + (K \times A) \times (K \times A)^* \arrow[l, bend left, "\mathsf{in}"] \arrow[d, "id + id \times look\ k"] \\
Maybe\ A & 1 + (K \times A) \times Maybe\ A \arrow[l, bend left, "g\ k"]
\end{tikzcd}
$$

Por inspecção do código apuramos $g\ k = [\underline{\mathsf{Nothing}}, g_2\ k]$, para

$$
\begin{array}{l}
g_2\ k\ ((a,b),c) \\
\quad \mid a \equiv k = \mathsf{Just}\ b \\
\quad \mid \mathsf{otherwise} = c
\end{array}
$$

Fazendo $d = (a,b)$, teremos:

$$
\begin{array}{l}
g_2\ k\ (d,c) \\
\quad \mid \pi_1\ d \equiv k = \mathsf{Just}\ (\pi_2\ d) \\
\quad \mid \mathsf{otherwise} = c
\end{array}
$$

Fazendo $x = (d,c)$, teremos:

$$
\begin{array}{l}
g_2\ k\ x \\
\quad \mid \pi_1\ (\pi_1\ x) \equiv k = \mathsf{Just}\ (\pi_2\ (\pi_1\ x)) \\
\quad \mid \mathsf{otherwise} = \pi_2\ x
\end{array}
$$

Passando a *poinfree*, teremos

$$g_2\ k = (\equiv k) \cdot (\pi_1 \cdot \pi_1) \rightarrow \mathsf{Just} \cdot \pi_2 \cdot \pi_1\ ,\ \pi_2$$

□

Questão 3 A função $\mathsf{foldr}\ \overline{\pi_2}\ i$ é necessariamente uma função constante. Qual? Justifique com o respectivo cálculo. **NB**: relembra-se a definição $\mathsf{foldr}\ f\ a = (\!|\, [\underline{a}, \widehat{f}] \,|\!)$.

RESOLUÇÃO: Vamos procurar k tal que $\underline{k} = \mathsf{foldr}\ \overline{\pi_2}\ i$ (justificar os passos que são dados):

$$
\begin{array}{ll}
 & \underline{k} = \mathsf{foldr}\ \overline{\pi_2}\ i \\
\equiv & \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \}
\end{array}
$$

$$\underline{k} = (\!| [\underline{i}, \pi_2] |\!)$$

$\equiv \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \}$

$$\begin{cases} \underline{k} \cdot \mathsf{nil} = \underline{i} \\ \underline{k} \cdot \mathsf{cons} = \pi_2 \cdot (id \times \underline{k}) \end{cases}$$

$\equiv \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \}$

$$\begin{cases} \underline{k} = \underline{i} \\ \underline{k} = \underline{k} \cdot \pi_2 \end{cases}$$

$\equiv \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \}$

$$k = i$$

Logo foldr $\overline{\pi_2}\ i = \underline{i}$, o que faz sentido pois em cada iteração o foldr seleciona sempre o resultado da cauda, terminando inevitavelmente no caso de base i.

$\square$

Questão 4 Uma árvore genealógica (T)

$$\mathbf{data}\ \mathsf{T}\ a = I\ \{ i :: a, m :: \mathit{Maybe}\ (\mathsf{T}\ a), p :: \mathit{Maybe}\ (\mathsf{T}\ a) \}$$

contém indivíduos (I) e os seus dois progenitores, caso sejam conhecidos (cf. Maybe). Propondo-se, para este tipo,

$$\mathsf{B}\ (X, Y) = X \times ((1 + Y) \times (1 + Y)) \tag{E2}$$

infira

$$\mathsf{in} : \mathsf{B}\ (X, \mathsf{T}\ X) \to \mathsf{T}\ X \tag{E3}$$

e defina, como um T-catamorfismo, a função

$$\mathit{indivs} : \mathsf{T}\ X \to X^*$$

que lista todos os indivíduos da sua árvore argumento. (Acompanhe a sua resposta com um diagrama.)

RESOLUÇÃO: Como o functor de base é dado, temos

$$\mathsf{B}\ (f, g) = f \times ((1 + g) \times (1 + g))$$
$$\mathsf{F}\ f = \mathsf{B}\ (id, f) = id \times ((id + f) \times (id + f))$$

e assim o diagrama:

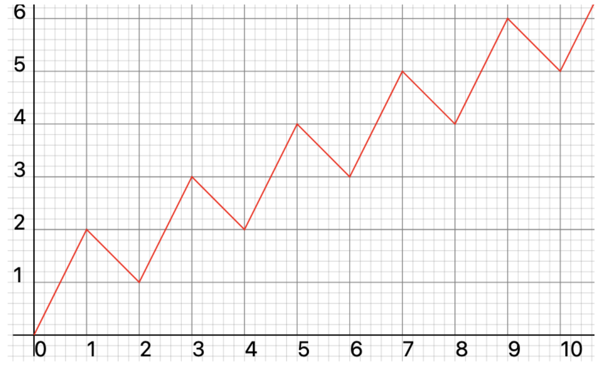
$$\begin{tikzcd}
\mathsf{T}\ X \arrow[d, "indivs = (\!| f |\!)"'] \arrow[r, bend left, "\mathsf{out} = \mathsf{in}^\circ"] \arrow[r, phantom, "\cong"] & X \times ((1 + \mathsf{T}\ X) \times (1 + \mathsf{T}\ X)) \arrow[l, bend left, "\mathsf{in}"] \arrow[d, "id \times ((id + indivs) \times (id + indivs))"] \\
X^* & X \times ((1 + X^*) \times (1 + X^*)) \arrow[l, bend left, "f"]
\end{tikzcd}$$

Vamos definir $1 + X^* \xrightarrow{h = [\mathsf{nil}, id]} X^*$ e usá-la no gene que usaremos para reunir todos os indivíduos,

$$f\ (x, (a, b)) = x : (h\ a +\!\!+\ h\ b)$$

isto é, $f = \mathsf{cons} \cdot (id \times (\mathsf{conc} \cdot (h \times h)))$. Ficam subentendidos in e out, que não são pedidos. $\square$

Questão 5 O gráfico mostra uma função h definida por recursividade mútua da forma seguinte,



$$\left\{\begin{array}{l} h = \mathsf{in} \cdot \mathsf{F}\ g \cdot \mathsf{out} \\ g = [\underline{1}, id] \cdot \mathsf{F}\ h \cdot \mathsf{out} \end{array}\right.$$

onde:

$$\begin{array}{l} \mathsf{F}\ f = id + f \\ \mathsf{in} = [\underline{0}, \mathsf{succ}] \\ \mathsf{succ}\ x = x + 1 \\ \mathsf{out} = \mathsf{in}^{\circ} \end{array}$$

Mostre que a mesma função pode ser definida à custa de um ciclo-for:

$$\begin{array}{l} h = \pi_1 \cdot \mathsf{for}\ loop\ (0,1)\ \mathbf{where} \\ \quad loop\ (a,b) = (1+b, a) \end{array}$$

RESOLUÇÃO: Como h e g estão em recursividade mútua, vamos usar essa lei (justificar os passos que são dados):

$$\begin{array}{cl}
& \left\{\begin{array}{l} h = \mathsf{in} \cdot \mathsf{F}\ g \cdot \mathsf{out} \\ g = [\underline{1}, id] \cdot \mathsf{F}\ h \cdot \mathsf{out} \end{array}\right. \\
\equiv & \{ \ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\ \} \\
& \left\{\begin{array}{l} h \cdot \mathsf{in} = \mathsf{in} \cdot \mathsf{F}\ g \\ g \cdot \mathsf{in} = [\underline{1}, id] \cdot \mathsf{F}\ h \end{array}\right. \\
\equiv & \{ \ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\ \} \\
& \left\{\begin{array}{l} h \cdot \mathsf{in} = \mathsf{in} \cdot \mathsf{F}\ (\pi_2 \cdot \langle h, g\rangle) \\ g \cdot \mathsf{in} = [\underline{1}, id] \cdot \mathsf{F}\ (\pi_1 \cdot \langle h, g\rangle) \end{array}\right. \\
\equiv & \{ \ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\ \} \\
& \langle h, g\rangle = \llparenthesis\, \langle \mathsf{in} \cdot \mathsf{F}\ \pi_2, [\underline{1}, id] \cdot \mathsf{F}\ \pi_1 \rangle \,\rrparenthesis \\
\equiv & \{ \ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\ \} \\
& \langle h, g\rangle = \llparenthesis\, \langle [\underline{0}, \mathsf{succ} \cdot \pi_2], [\underline{1}, \pi_1] \rangle \,\rrparenthesis \\
\equiv & \{ \ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\ \} \\
& \langle h, g\rangle = \llparenthesis\, [\underline{(0,1)}, \langle \mathsf{succ} \cdot \pi_2, \pi_1 \rangle] \,\rrparenthesis \\
\equiv & \{ \ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\ \} \\
& \langle h, g\rangle = \mathsf{for}\ \langle \mathsf{succ} \cdot \pi_2, \pi_1 \rangle\ (0,1) \\
\equiv & \{ \ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\ \} \\
& \langle h, g\rangle = \mathsf{for}\ loop\ (0,1)\ \mathbf{where}\ loop\ (a,b) = (1+b, a) \\
\equiv & \{ \ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\ \} \\
& \left\{\begin{array}{l} h = \pi_1 \cdot (\mathsf{for}\ loop\ (0,1)\ \mathbf{where}\ loop\ (a,b) = (1+b,a)) \\ g = \pi_2 \cdot (\mathsf{for}\ loop\ (0,1)\ \mathbf{where}\ loop\ (a,b) = (1+b,a)) \end{array}\right.
\end{array}$$

□

Questão 6 Nas aulas foi demonstrada a seguinte lei dos catamorfismos:

$$(\!|\, g \,|\!) \cdot (\!|\, \mathsf{in}_2 \cdot \alpha \,|\!) = (\!|\, g \cdot \alpha \,|\!) \quad \Leftarrow \quad \mathsf{G}\, f \cdot \alpha = \alpha \cdot \mathsf{F}\, f$$

Demonstre agora a correspondente lei dos anamorfismos,

$$[\!(\alpha \cdot \mathsf{out}_2)\!] \cdot [\!(g)\!] = [\!(\alpha \cdot g)\!] \quad \Leftarrow \quad \alpha \cdot \mathsf{G}\, f = \mathsf{F}\, f \cdot \alpha \tag{E4}$$

a que corresponde o diagrama:

$$\begin{tikzcd}
\mathsf{T}_1 \arrow[rr, "\mathsf{out}_1"] & & \mathsf{F}\ \mathsf{T}_1 \\
\mathsf{T}_2 \arrow[u, "{[\!(\alpha \cdot \mathsf{out}_2)\!]}"] \arrow[r, "\mathsf{out}_2"] & \mathsf{G}\ \mathsf{T}_2 \arrow[r, "\alpha"] & \mathsf{F}\ \mathsf{T}_2 \arrow[u, "{\mathsf{F}\ [\!(\alpha \cdot \mathsf{out}_2)\!]}"'] \\
C \arrow[u, "{[\!(g)\!]}"] \arrow[r, "g"'] & \mathsf{G}\ C \arrow[u, "{\mathsf{G}\ [\!(g)\!]}"'] \arrow[r, "\alpha"'] & \mathsf{F}C \arrow[u, "{\mathsf{F}[\!(g)\!]}"']
\end{tikzcd} \tag{E5}$$

Sugestão: recorra à lei de fusão-ana, entre outras que conhece.

RESOLUÇÃO: Tem-se (preencher as justificações):

$$\begin{array}{ll}
 & [\!(\alpha \cdot \mathsf{out}_2)\!] \cdot [\!(g)\!] = [\!(\alpha \cdot g)\!] \\
\Leftarrow & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
 & \alpha \cdot \mathsf{out}_2 \cdot [\!(g)\!] = \mathsf{F}\ [\!(g)\!] \cdot \alpha \cdot g \\
\equiv & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
 & \alpha \cdot \mathsf{G}\ [\!(g)\!] \cdot g = \mathsf{F}\ [\!(g)\!] \cdot \alpha \cdot g \\
\Leftarrow & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
 & \alpha \cdot \mathsf{G}\ [\!(g)\!] = \mathsf{F}\ [\!(g)\!] \cdot \alpha \\
\Leftarrow & \quad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
 & \alpha \cdot \mathsf{G}\, f = \mathsf{F}\, f \cdot \alpha
\end{array}$$

$\square$

Questão 7 A função factorial

$$\begin{array}{l}
\mathit{fac}\ 0 = 1 \\
\mathit{fac}\ (n+1) = (n+1) * \mathit{fac}\ n
\end{array}$$

pode ser escrita sob a forma

$$\mathit{fac} \cdot \mathsf{in} = g \cdot \mathsf{F}\ \langle \mathit{id}, \mathit{fac} \rangle \tag{E6}$$

onde

$$\begin{array}{l}
\mathsf{in} = [\mathsf{zero}, \mathsf{succ}] \\
g = [\mathsf{one}, \mathsf{mul} \cdot (\mathsf{succ} \times \mathit{id})] \\
\left\{ \begin{array}{l} \mathsf{F}\ X = 1 + X \\ \mathsf{F}\ f = \mathit{id} + f \end{array} \right.
\end{array}$$

Sendo dado

$$\left\{ \begin{array}{l} \mathsf{G}\ X = \mathsf{F}\ (\mathbb{N}_0 \times X) = 1 + \mathbb{N}_0 \times X \\ \mathsf{G}\ f = \mathit{id} + \mathit{id} \times f \end{array} \right.$$

complete as reticências no seguinte processo de transformação de (E6) num **hilomorfismo** de listas, desenhando depois o respectivo **diagrama**:

$$
\begin{array}{ll}
& fac \cdot \mathsf{in} = g \cdot \mathsf{F}\,\langle id, fac\rangle \\
\equiv & \qquad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& fac = g \cdot \mathsf{F}\,\langle id, fac\rangle \cdot \mathsf{out} \\
\equiv & \qquad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& fac = g \cdot \mathsf{F}\,((id \times fac) \cdot \langle id, id\rangle) \cdot \mathsf{out} \\
\equiv & \qquad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& fac = g \cdot \mathsf{F}\,(id \times fac) \cdot \mathsf{F}\,\langle id, id\rangle \cdot \mathsf{out} \\
\equiv & \qquad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& fac = g \cdot \mathsf{G}\,fac \cdot \mathsf{F}\,\langle id, id\rangle \cdot \mathsf{out} \\
\equiv & \qquad \{ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \} \\
& fac = (\!|\, g \,|\!) \cdot [\!(\mathsf{F}\,\langle id, id\rangle \cdot \mathsf{out})\!]
\end{array}
$$

RESOLUÇÃO: Com as justificações:

$$
\begin{array}{ll}
& fac \cdot \mathsf{in} = g \cdot \mathsf{F}\,\langle id, fac\rangle \\
\equiv & \qquad \{ \text{ `Shunt-left' para isomorfismo in/out } \} \\
& fac = g \cdot \mathsf{F}\,\langle id, fac\rangle \cdot \mathsf{out} \\
\equiv & \qquad \{ \text{ natural-}id\text{ duas vezes; absorção-}\times \} \\
& fac = g \cdot \mathsf{F}\,((id \times fac) \cdot \langle id, id\rangle) \cdot \mathsf{out} \\
\equiv & \qquad \{ \text{ functor-F } \} \\
& fac = g \cdot \mathsf{F}\,(id \times fac) \cdot \mathsf{F}\,\langle id, id\rangle \cdot \mathsf{out} \\
\equiv & \qquad \{ \ \mathsf{G}\,f = \mathsf{F}\,(id \times f) \ \} \\
& fac = g \cdot \mathsf{G}\,fac \cdot \mathsf{F}\,\langle id, id\rangle \cdot \mathsf{out} \\
\equiv & \qquad \{ \text{ hilo-factorização } [\![f, g]\!] = (\!|\, f \,|\!) \cdot [\!(g)\!] \ \} \\
& fac = (\!|\, g \,|\!) \cdot [\!(\mathsf{F}\,\langle id, id\rangle \cdot \mathsf{out})\!]
\end{array}
$$

$\square$

Questão 8 Recordando o mónade das listas,

$$X \xrightarrow{\ \mathsf{singl}\ } X^* \xleftarrow{\ \mathsf{concat}\ } (X^*)^* \tag{E7}$$

mostre que a função de filtragem de uma lista por um predicado p,

$$\mathsf{filter}\ p = (p \rightarrow \mathsf{return}, \mathsf{nil}) \bullet id \tag{E8}$$

é um catamorfismo de listas, sabendo que $\mathsf{concat} = (\!|\, [\mathsf{nil}, \mathsf{conc}] \,|\!)$, $\mathsf{nil}\ \_ = [\,]$, $\mathsf{singl}\ x = [x]$ e $\mathsf{conc} = \widehat{(+\!\!+)}$. De seguida, derive a correspondente versão *pointwise*.

RESOLUÇÃO: Para filter p ser um catamorfismo é preciso encontrar g tal que $(\!| g |\!) = \mathsf{filter}\ p$. Vamos resolver essa equação (preencher as justificações):

$$
\begin{array}{ll}
& (\!| g |\!) = \mathsf{filter}\ p \\
\equiv & \quad \{ \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad \} \\
& (\!| g |\!) = \mathsf{concat} \cdot \mathsf{map}\ (p \rightarrow \mathsf{return}, \mathsf{nil}) \cdot id \\
& \quad \{ \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad \} \\
& (\!| g |\!) = (\!| [\mathsf{nil}, \mathsf{conc}] |\!) \cdot \mathsf{map}\ (p \rightarrow \mathsf{return}, \mathsf{nil}) \\
& \quad \{ \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad \} \\
& (\!| g |\!) = (\!| [\mathsf{nil}, \mathsf{conc}] \cdot (id + (p \rightarrow \mathsf{return}, \mathsf{nil}) \times id) |\!) \\
& \quad \{ \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad \} \\
& (\!| g |\!) = (\!| [\mathsf{nil}, \mathsf{conc} \cdot ((p \rightarrow \mathsf{return}, \mathsf{nil}) \times id)] |\!)
\end{array}
$$

Logo, filter p é o catamorfismo $(\!| [\mathsf{nil}, \mathsf{conc} \cdot ((p \rightarrow \mathsf{return}, \mathsf{nil}) \times id)] |\!)$.
$\square$
