On the *'A'* that links the *'M'*s of Maths, Music and Maps

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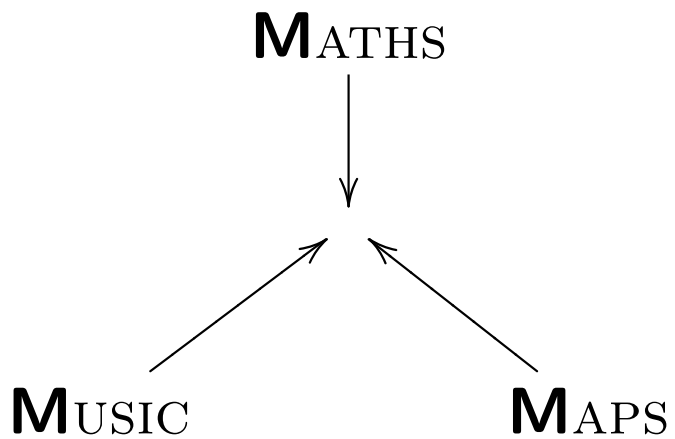
The "3M rule"

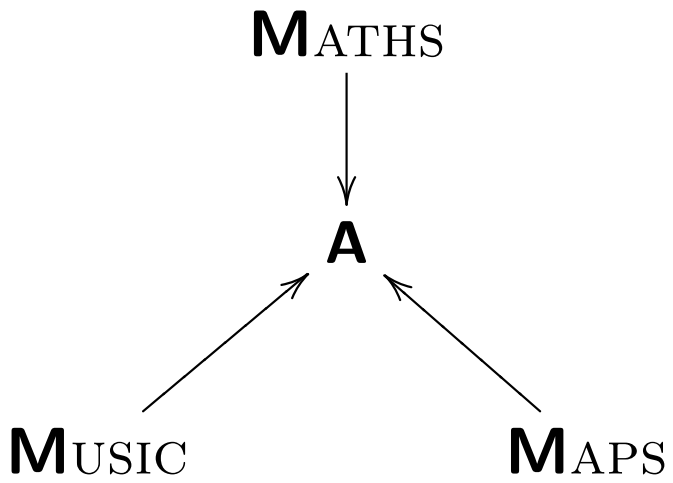
Someone has observed that *"those who like maths also enjoy music and maps"* [1]:



What commonalities can be found between such a language system (Maths), an art (music) and a science (cartography)?

[1] Comment by a student of David Naumann, Stevens Institute, New Jersey.

The "3M rule"



Maths is an *abstract* language (**A**)
Music is perhaps the most *abstract* (**A**) of all arts
Maps are geographical *abstractions* (**A**)

Abstraction

Quoting Jeff Kramer [2]:

*"**Abstraction** is widely used in other disciplines such as **art** and **music**. For instance (...) Henri Matisse manages to clearly represent the **essence** of his subject, a naked woman, using only simple lines or cutouts. His representation **removes** all detail yet **conveys** much."*

[2] *Is Abstraction the Key to Computing?*, Commun. ACM, 50:4, pages 37–42, April 2007.

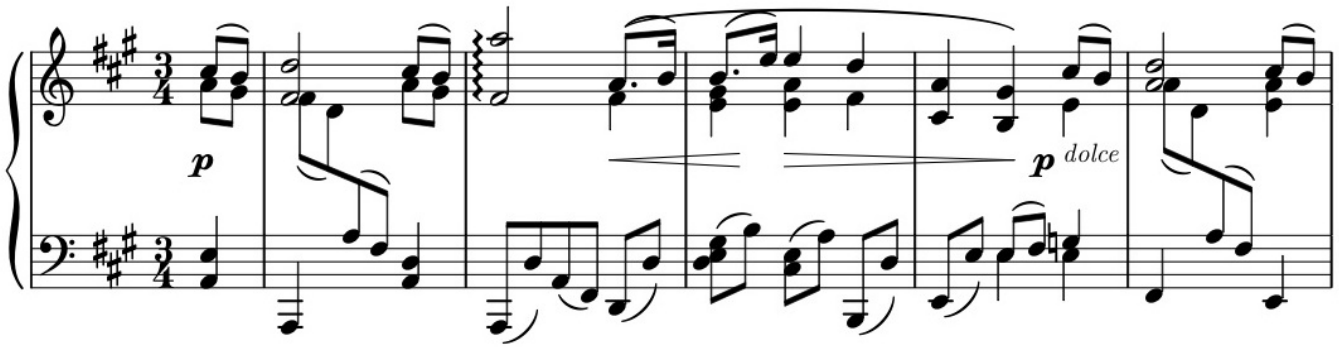
Abstraction

The famous "abstract" map of the London Underground (1939)

BASE PRINCIPLE:
"Eliminate characteristics of the mapped object that are **not relevant** *to the map's* **purpose***"*

Abstraction

Opus 118, no.2, by Johannes Brahms (1833-97):



Questions:

> *What does this piece* **mean***? Does it* **describe** *anything? Does it* **imitate** *or recreate reality?*

Answer:

> *It means* **nothing** *— it is* **abstract***!*

Leonard Bernstein (1st *Young People's Concerts*, 18-Jan-1958):

> *"Music is never about anything: music just 'is'!"*

Abstraction

Albert Camus (1913-60):

> *Music is the* **perfect expression** *of an* **ideal world** *which is communicated to us through harmony. This world exists. Not at a level higher or lower than the real world, but parallel to it.*
>
> **World of ideas***? Maybe. Or else* **world of numbers**, *as communicated to us by Harmony."*

(Essay on Music, 1932)

Maths

Mathematics is the **universal** language of science. Why?

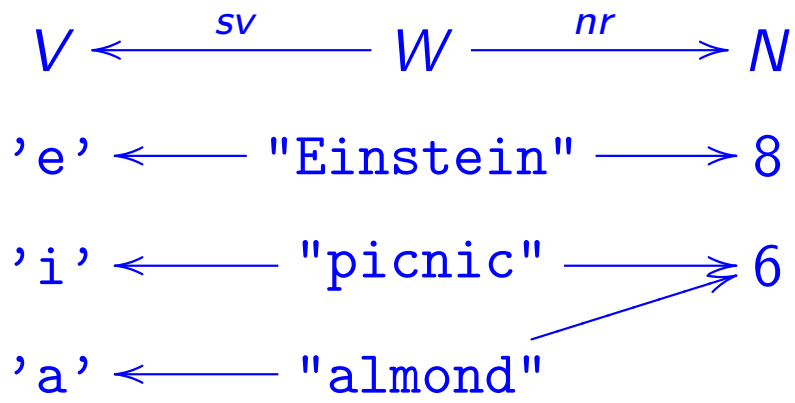
- The **abstract** language par excellence
- Safe (unambiguous) means to pass knowledge between generations
- Mathematical **proof** — the ideal way to provide verifiable **evidence**.

Abstraction: what is it, after all?

Our answer will be based on a concept of mathematics itself — that of a **function**.

Two functions

Number of letters (nr) and stress vowel (sv)



Notation (Leibniz): $6 = nr(\texttt{"almond"})$, $\texttt{'a'} = sv(\texttt{"almond"})$, etc

In general:

$$y = f(x)$$

Functional abstraction

Given some function $A \xrightarrow{f} B$ such that, for all b in B, there is some a from A such that $b = f(a)$, for example

$$W \xrightarrow{sv} V$$

$$\texttt{"almond"} \longrightarrow \texttt{'a'} \qquad (etc)$$

$$\texttt{"Einstein"} \longrightarrow \texttt{'e'} \qquad (etc)$$

$$\texttt{"picnic"} \longrightarrow \texttt{'i'} \qquad (etc)$$

$$\texttt{"poll"} \longrightarrow \texttt{'o'} \qquad (etc)$$

$$\texttt{"future"} \longrightarrow \texttt{'u'} \qquad (etc)$$

we say that domain B is **more abstract** than A and that f is a **witness** of such an abstraction.

In the example: **one** stress Vowel abstracts **many** Words.

Textual abstraction functions

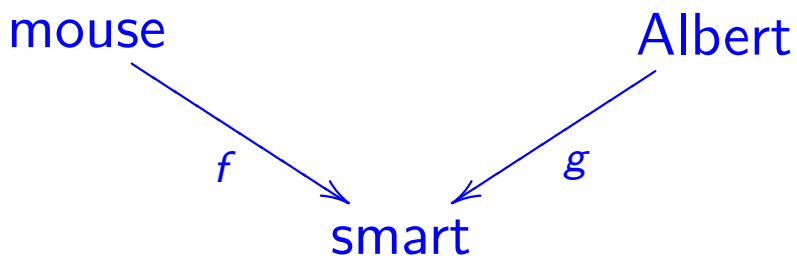
(a)	As Albert is as smart as a mouse and fast as a hare...	... he fled without a trace
	$\downarrow f$	
(b)	Albert ... smart ... mouse ... fast ... hare...	... fled without a trace
	$\downarrow g$	
(c)	Mousy 'Mr. Hare' ...	... fled without a trace

We see that by losing textual information, the text acquires a **metaphorical** dimension — e.g. nickname 'Hare', and so on.

"A la Chomsky": functions f and g transform **deep** *structure* (a) into **surface** *structure* (c).

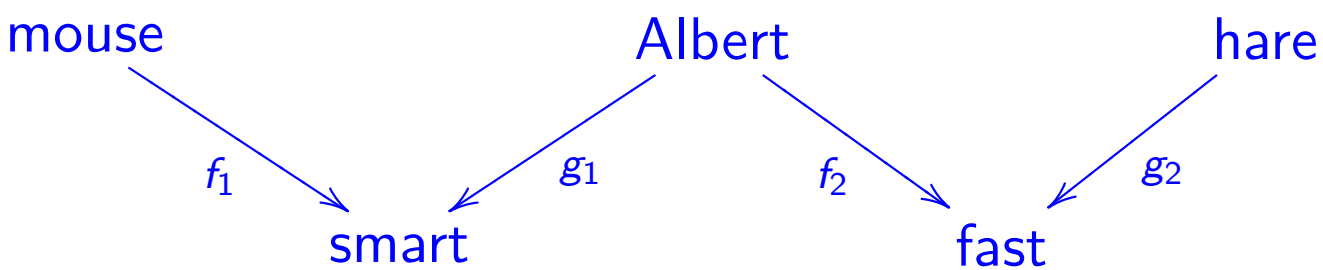
How two functions make a metaphor

Example:



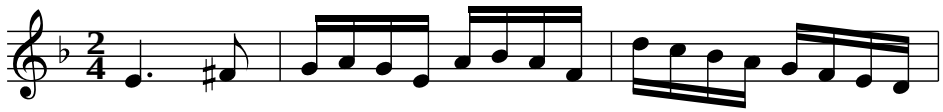
where f and g are the **witnesses** of the metaphor.

In the example there are two juxtaposed metaphors:

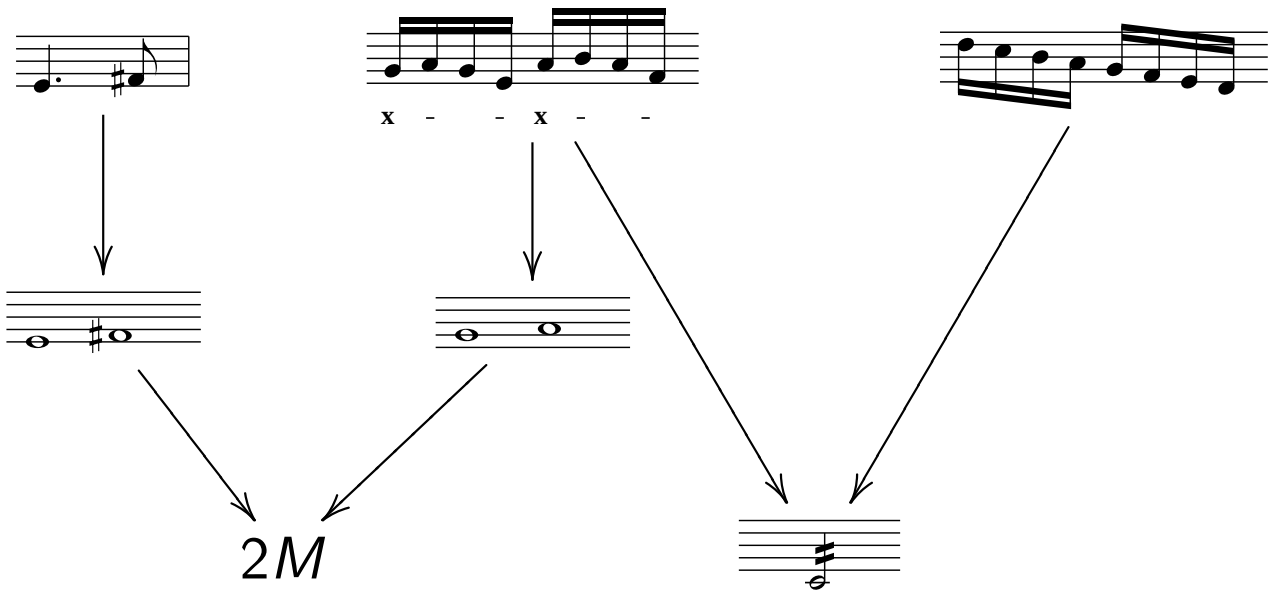


Metaphors in music

Listen to the music excerpt



which includes two simple metaphors, one **melodic** and the other **rhythmic**:

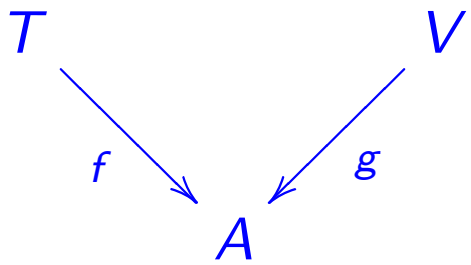


Metaphors as maths objects

A **metaphor** can be regarded as a "cospan" (aside) in which

- T (*tenor*) is the subject
- V is the *vehicle*
- A is the shared *attribute*.

(Cf. Richards' *Philosophy of Rhetoric*, 1936).



Summing up, a metaphor is a **binary relationship**

$$T(f^{\circ} \cdot g)V \tag{1}$$

in which the attribute (A) is hidden.

Metaphors as maths objects

Brief explanation of the formula in the previous slide:

- R° denotes R in
  passive voice: $b\ R\ a$ thus means the same as $b\ R^\circ\ a$. Example:

  Albert watches the hare

  versus

  the hare is watched by Albert

- Composition: $(f \cdot g)x = f(g\ x)$, for instance

  $f\ p =$ *the number of letters of* p
  $g\ p =$ *the stressed syllable of* p

  Thus:

  $(f \cdot g)p =$ *the number of letters of the stressed syllable of* p

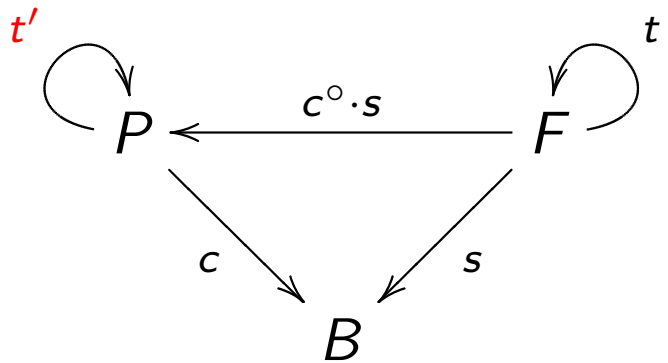
Putting the **vehicle** in motion

'No jobs for the boys' in metaphorical form, quoting Eça de Queirós (1845-1900):[3]

"Os políticos e as fraldas devem trocar-se frequentemente e pela mesma razão"

"Politicians and diapers should be changed often and for the same reason"

Metaphor:

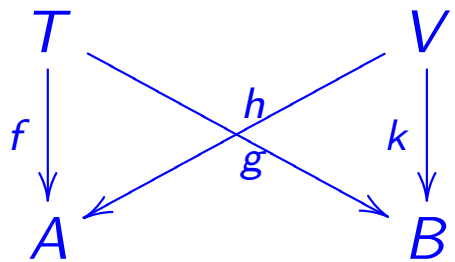


Axiom: $s\ (t\ x) = False$ — inducing *change* t' in P, etc etc.

[3] Abbreviations: P = politician (tenor); F = nappy (vehicle); c = *corrupted* ; s = *dirty* ; t = *change*.

Metaphors and ambiguity

In the presence of more than one shared attribute, e.g.



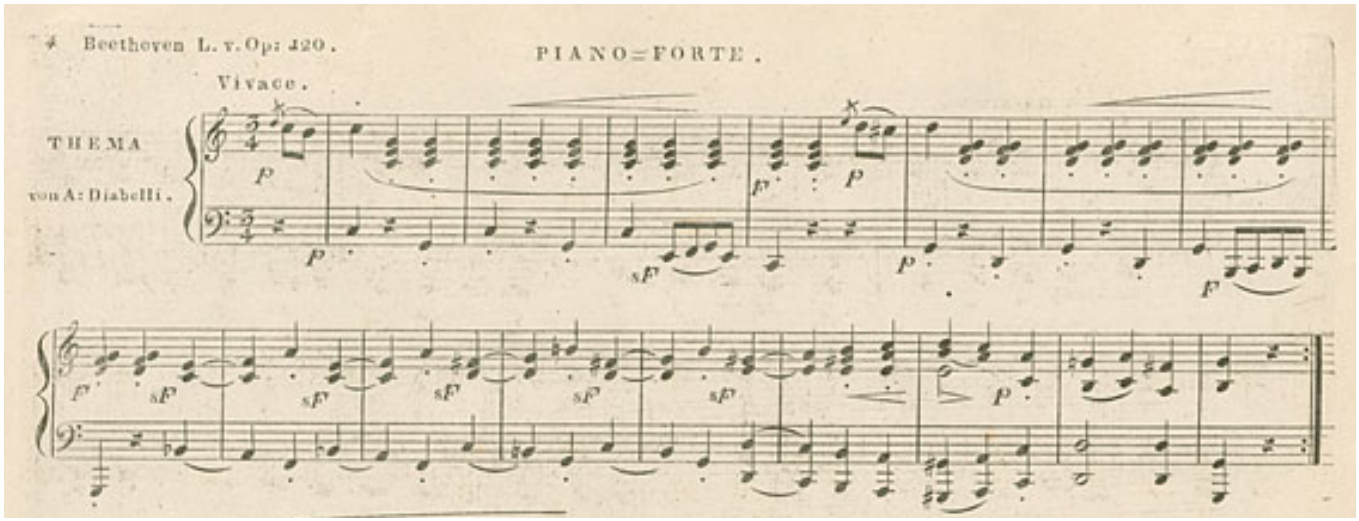
attribute omission leads to **ambiguity**.

Either **context** suggests the attribute or the text becomes **open** and may acquire a **poetic** dimension.

Metaphors often "close" inter-textually — see next slide.

Metaphors in music

A well-known anecdote (Vienna, 1820s):



Vaterländischer Künstlerverein (by 51 composers) on a waltz by Anton Diabelli (1781-1858).

Var.XXII of contribution 51

A. Diabelli (1781-1858)

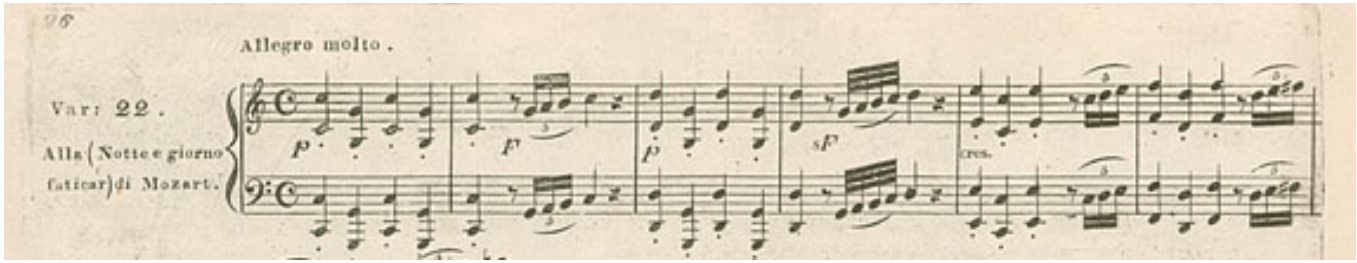
W.A. Mozart (1756-1791)

Don Giov.: Notte e giorno faticar (Fg,Vc,Ba)

L. van Beethoven (1770-1827)

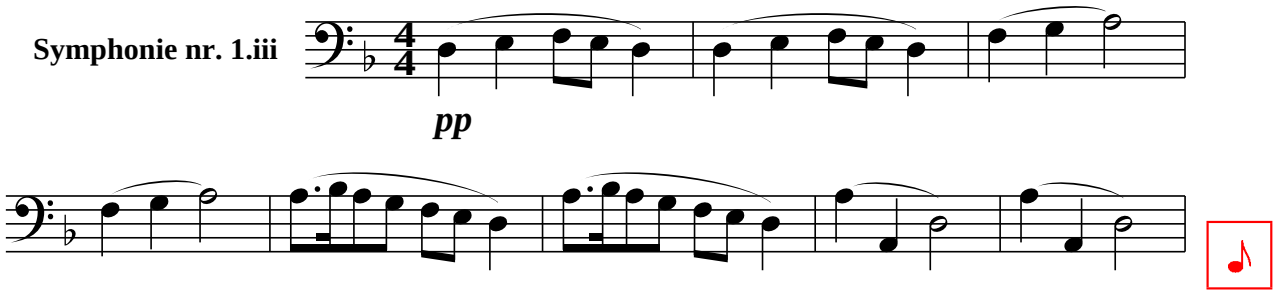
op120.xxii

etc

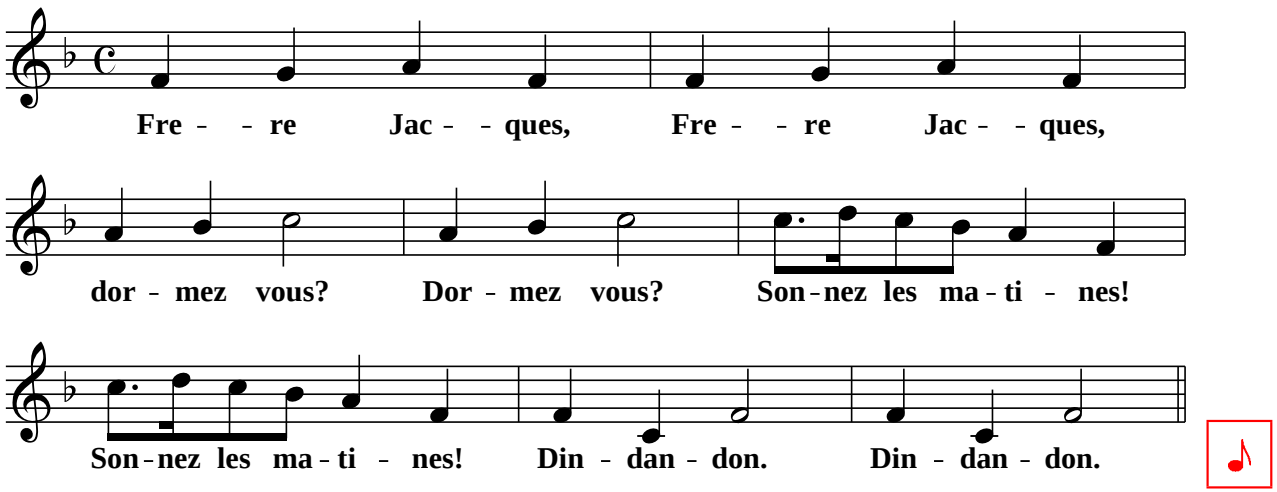


Metaphors in music

Funeral march by Gustav Mahler (1860-1911): metaphor with **tenor**

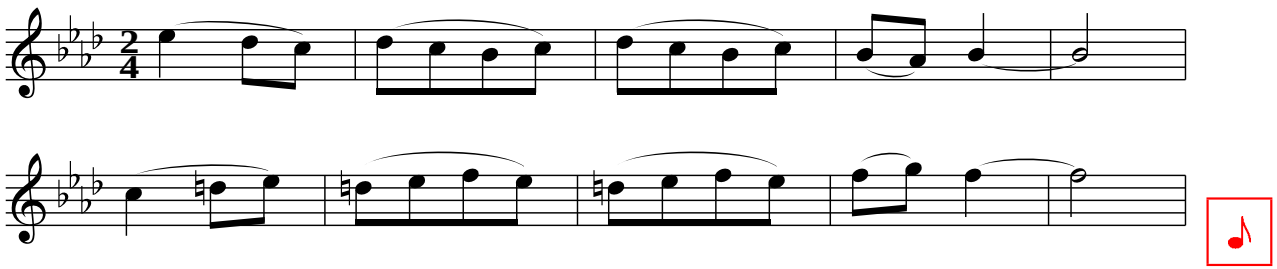


whose **vehicle** is the popular:



(More abstract) metaphors in music

3rd mov. of Brahms (1833-97) 1st symphony:



Sentence alone is metaphoric by itself, why?

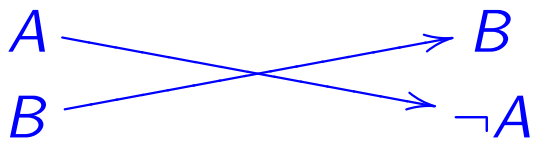
- Too obvious: two halves share the same **rythm**
- Less obvious: 2nd half (**tenor**) is an **inversion** of 1st half (**vehicle**).

Music can be very 'metaphoric' in this (rather **formal**) sense.

Chiasmus

José Saramago (1922-2010):

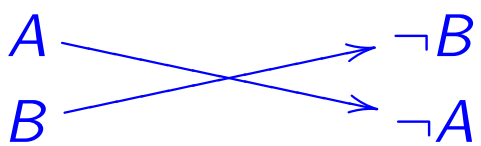
Deus está em mim,



ou em mim não está Deus

António Vieira (1608-97):

Os homens (tinham) a razão sem uso

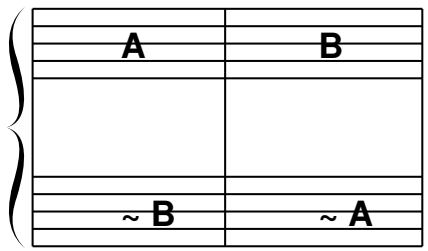


os peixes o uso sem a razão

(shaped like the letter χ, cf. root $\chi\alpha\sigma\mu\acute{o}\varsigma$ from the Greek)

Chiasmus in music

The sequence $A\ B\ \neg B\ \neg A$ of terms that cross each other in the *parallel* or *double antithesis* of a **chiasmus** becomes **parallel de facto** in music, thanks to **polyphony**:



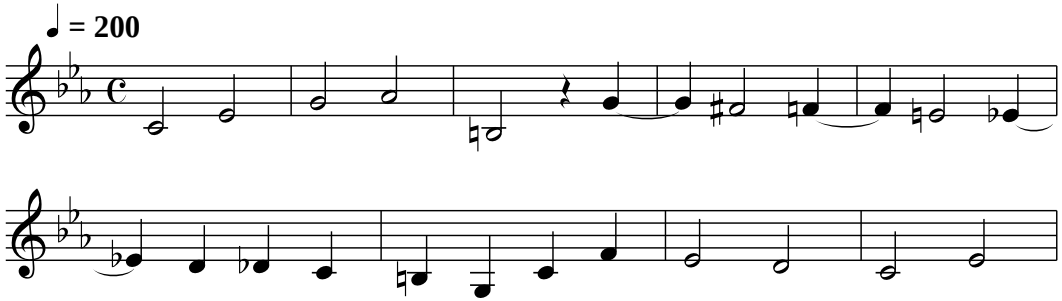
Antithesis by **retrograde** motion (cf. *algebraic* sequence inversion) exhibiting properties such as e.g.

$$\begin{aligned} \neg\neg A &= A \\ \neg(A\ B) &= (\neg B)\ (\neg A) \end{aligned}$$

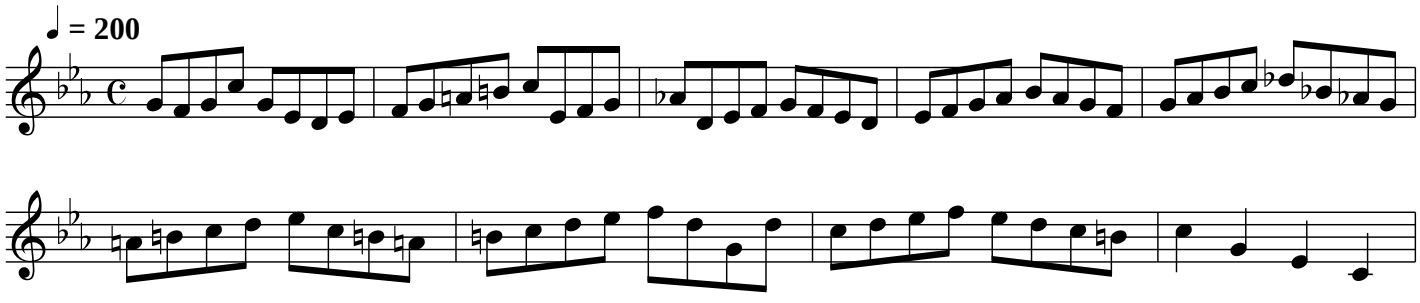
Chiasmus in music

Abundant device in e.g. Baroque music, cf. *Canon a 2 super thema regium* from BVW 1079 by J.S. Bach (1685-1750):

A =



B =

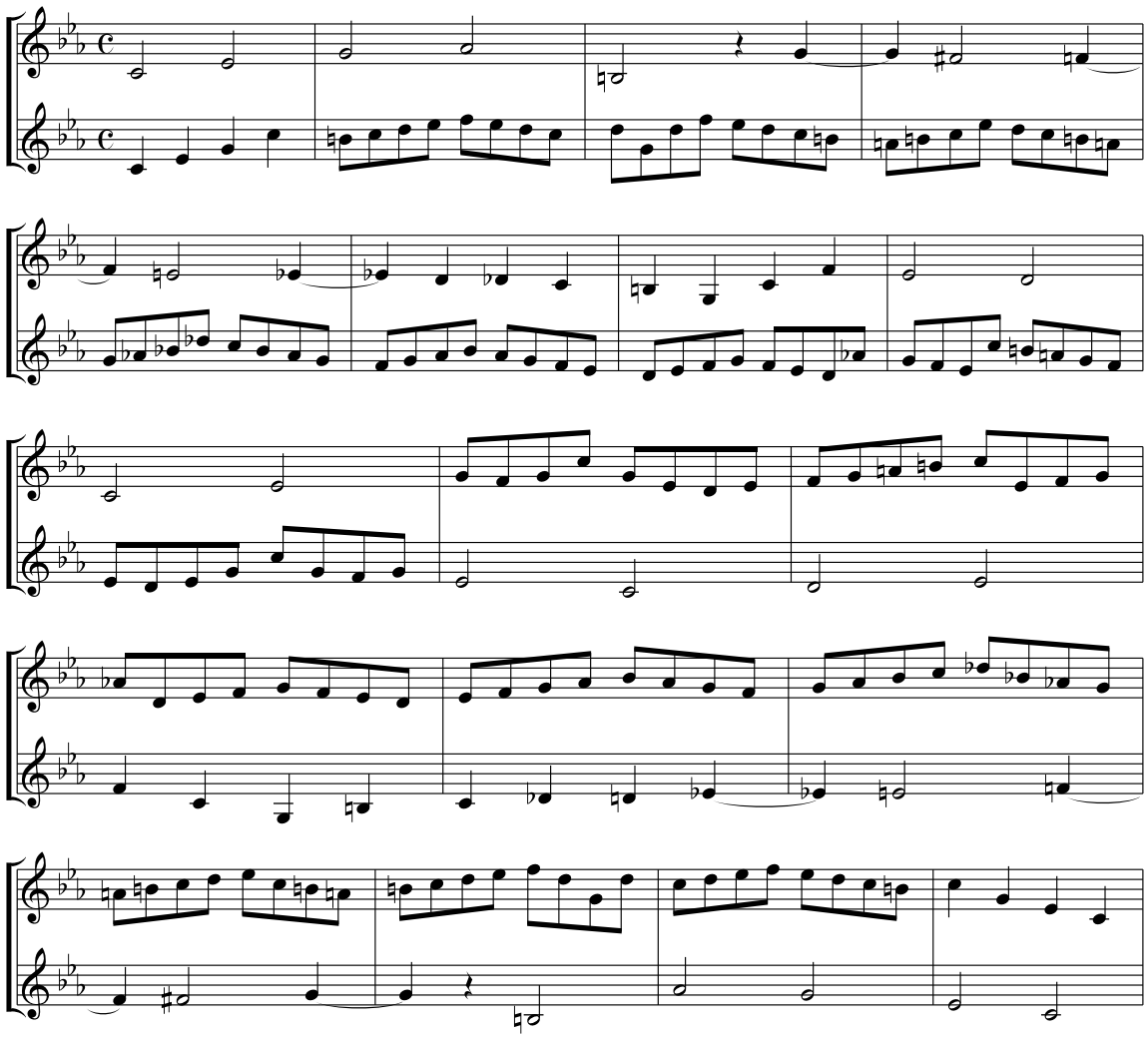


Musical Offering BWV 1079

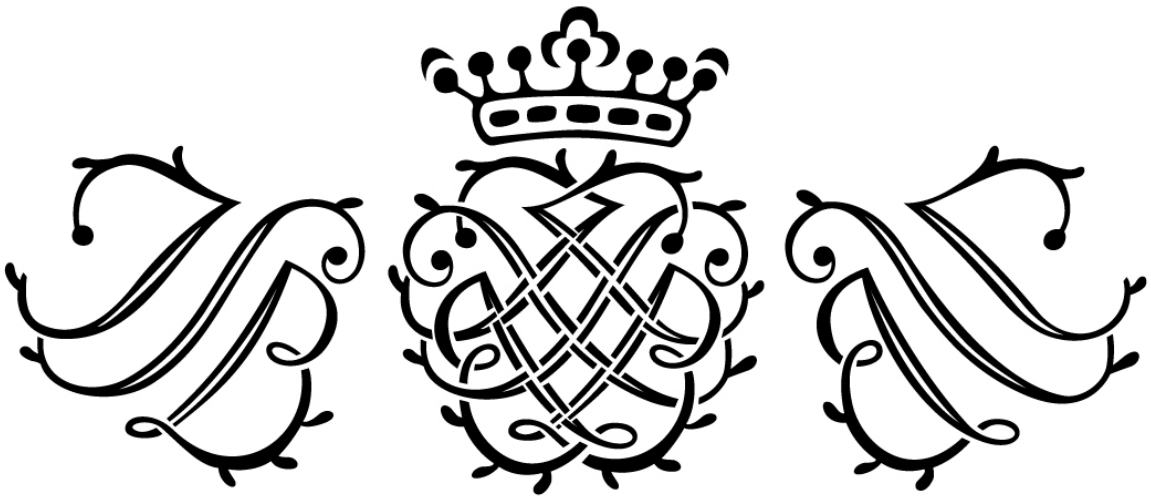
Canon a 2

(nr.1 of Canones diversi super thema regium, BWV 1079)

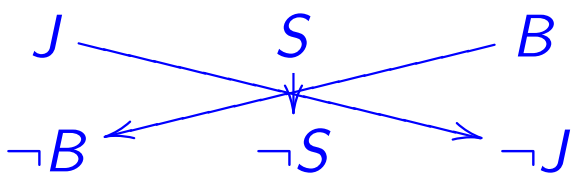
J.S. Bach (1685-1750)



Still about "chiasmus"



The stamp of J.S. Bach (1685-1750) in which the initials 'J', 'S', 'B' overlap each other once "mirrored" (ie. inverted), cf.



Algebraicity

Apparently a requirement "wilfully" claimed by Saussure in his writings (ELG, p. 236): [4]

> *"l'expression simple sera algébrique ou ne sera pas"*

In what measure can universal **algebra** help?

Music particularly interesting in this respect, for its inherent **algebraicity**:

- Music event level forms a very simple **algebra**: that of **sequences** of pairs (*pitch*, *duration*).
- Transformations across musical metaphor witnesses easy to express and reason about (many can be regarded as **linear transforms**).

[4] S. Bouquet, *Ontologie et Épistemologie de la Linguistique dans les Textes Originaux de Ferdinand de Saussure*, U. Paris X, 2008, vol. XVIII, no.3.

Algebraicity

Take **augmentation**, for instance:

$$k * [\,] = [\,]$$
$$k * [(p, d)] = [(p, k\ d)]$$
$$k * (m\ n) = (k * m)\ (k * n)$$

Transposition:

$$[\,] + i = [\,]$$
$$[(p, d)] + i = [(p + i, d)]$$
$$(m\ n) + i = (m + i)\ (m + i)$$

Retrograde:

$$\neg[\,] = [\,]$$
$$\neg[(p, d)] = [(p, d)]$$
$$\neg(m\ n) = (\neg n)\ (\neg m)$$

etc

Algebraicity

In general, can one "measure" algebraically the **expressive richness** of a metaphor $f^\circ \cdot g$?

Algebra of functions: $f \leqslant g$ measuring loss of information — f **abstracts** more than g ($\equiv f$ is less **injective** than g).

Distance between tenor and vehicle of metaphor $f^\circ \cdot g$ measurable by **complements** $\neg f$ and $\neg g$ ($\equiv$ what is **not** common), where $\neg f$ satisfies the universal property

$$id \leqslant f \vartriangle k \quad \equiv \quad \neg f \leqslant k$$

where id is the identity and $(f \vartriangle g)\ x = (f\ x, g\ x)$.

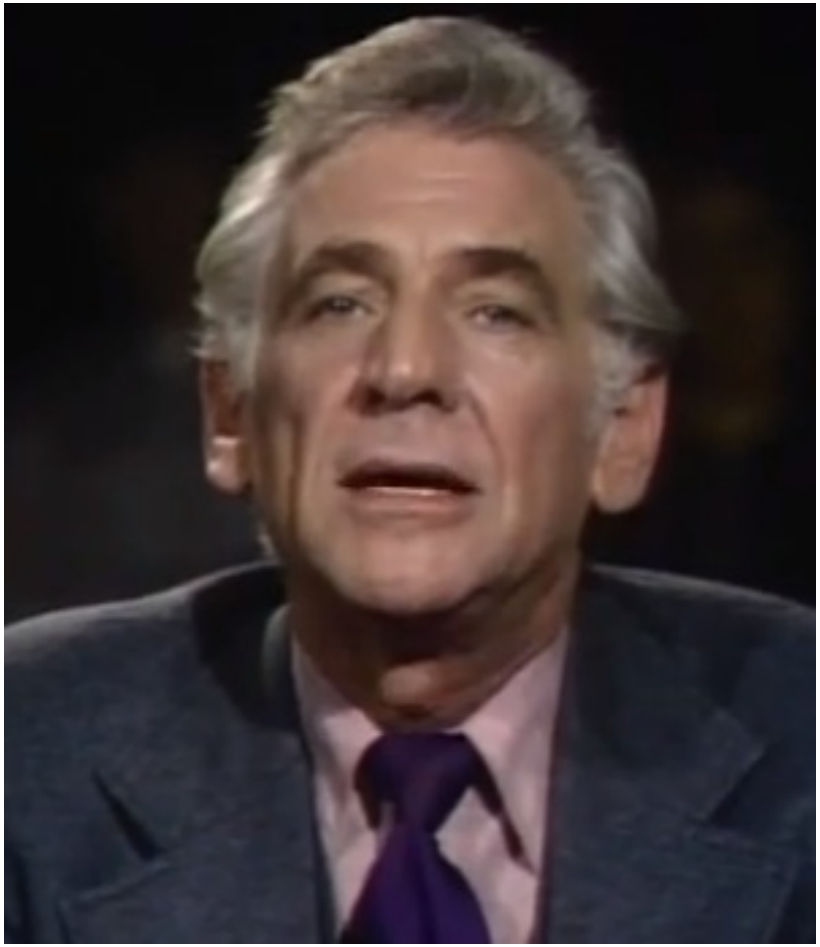
For example, small complements will correspond to *poor* metaphors, etc.

The Unanswered Question (Bernstein at Harvard)

"He immersed himself in Chomskyan **linguistics** *(...) so that he could then apply the principles of* **linguistics** *to* **music** *— thereby creating a brand new field of study.*

Ambitious? Oh, yes! Was he in over his head? Completely!"

(Jamie Bernstein).



The Unanswered Question (Bernstein at Harvard)

For Leonard Bernstein (1919-90), music is

- inherently **transformational**
- the **most metaphorical** of all means of artistic expression.

In fact:

- musical processing functions are easy to identify (cf. **algebra** of sequences);
- metaphors in music are **inherently abstract** — they close up literally on the musical text alone.

Expressive efficacy proportional to the **'metaphorical engineering'** of the composer — have a look at our last example (next slides).

Creative process

The example leads us back to Chomsky's (hidden) **deep structures** (prosaic, vulgar) as opposed to the (visible, audible) **surface structures** (poetic, beautiful).

In a sense, isn't this **deep-to-surface** transformation an essential part of the **creative** process itself?

Sketch books of L. van Beethoven (1770-1827) — 7000 manuscript folia (many on the internet[5]) ready for this kind of study.

Deep structure starting point[6]: a vulgar, 'Mannheim-like" theme (see next slide).

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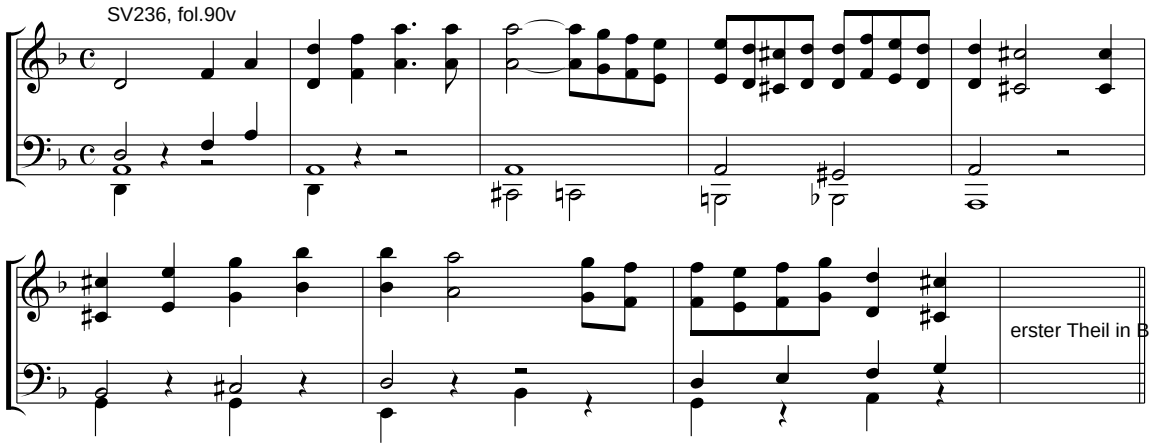
[5] See *Beethoven sketches in the Digital Archives* available from http://www.beethoven-haus-bonn.de.

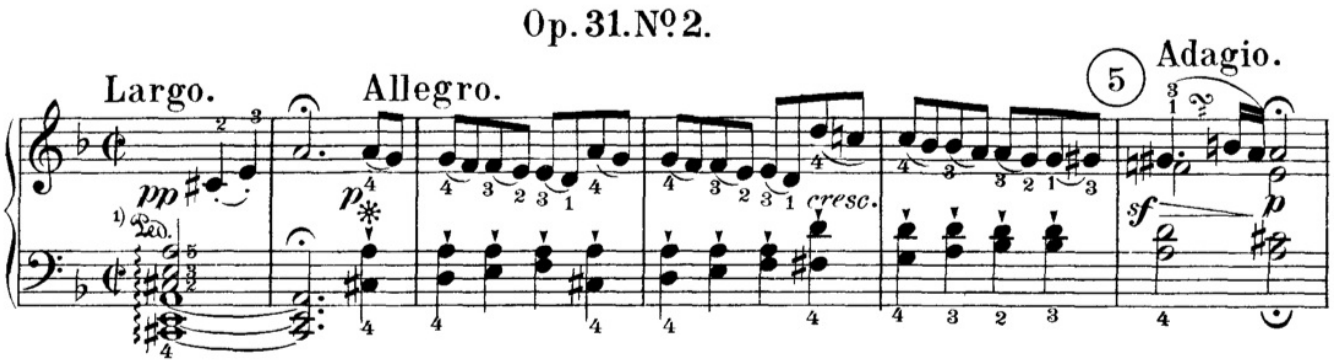
[6] Cf. B. Cooper, *Beethoven and the Creative Process*, Clarendon Press, 1990.

Creative process

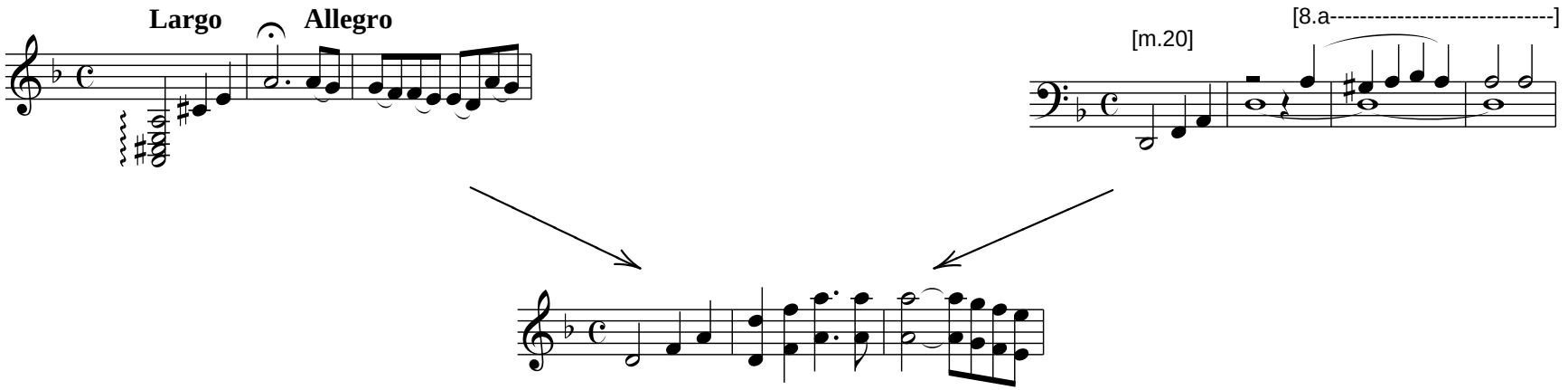
Deep structure



From this, sonata opus 31-no.2 eventually emerged:



Sketches unveil the metaphor



"Mannheim rocket" theme in the sketches is the hidden (common) attribute of the whole movement's metaphor.

NB: see chap. 12 of *Cooper*, *op.cit*, for a detailed study of the sketches of this movement of the sonata.

"Metaphors We Live By"

In their classic textbook, George Lakoff and Mark Johnson show how metaphorical "normal language" is in everyday life, eg.:

- ARGUMENT IS WAR (read: $\mathit{Argument} \xleftarrow{f^\circ \cdot g} \mathit{War}$ for some witnessing f, g)
- TIME IS MONEY (read: $\mathit{Time} \xleftarrow{f^\circ \cdot g} \mathit{Money}$ for some other witnesses f, g)

My question is:

> *Since our brain structures concepts and knowledge around so many "down-to-earth" metaphors, how do such metaphors acquire the* **æstetical** *drive which lifts us to Camus' (parallel)* **ideal world***?*

Such is the magic of a **creative mind**.

Towards the Semantics of Music

Mihailo Antovic[7] (Univ. Nis, Serbia):

> *Music is an* **abstraction**, *and the only way to approach it is to* **metaphorise** — *i.e. map the* **concrete** *onto the* **abstract**, *be it through waterfalls, bamboos or dots on a vertical line.*
>
> *(...) conceptual metaphor theory, in its search of the conceptualization of music, provides the most* **solid grounds** *for a true '***musico-semantics***'.*

[7] Antovic, M. (2009). Towards a Semantics of Music - the 20th Century, Language and History, 52(1): 119-129.

Afterthought

Two flavours in (applied) **linguistics**,

- **generative** (grammars, parsing ....)
- **cognitive** ("metaphors we live by"...)

Parallel in software science:

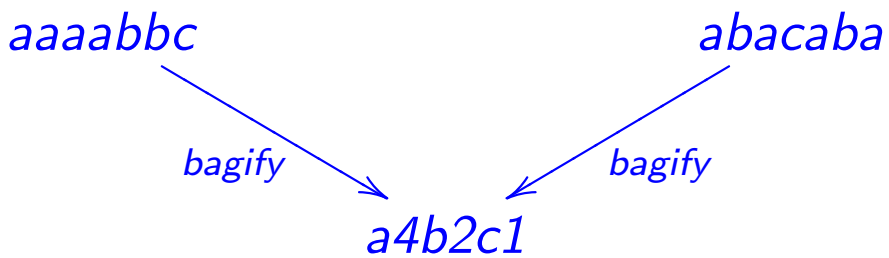
- "Hylomorphisms" with pattern $f \cdot g^{\circ}$, eg. contex-free compilers:

$$compiler = code\_generator \cdot pretty\_printer^{\circ}$$

- "Metaphorisms" with pattern $f^{\circ} \cdot g$, eg. sorting:

$$sort = is\_ordered \cdot (bagify^{\circ} \cdot bagify)$$

cf.



Epilogue

Humanities versus **Science**

- The *Big Divide*: "Art" or " Science" ... — a historical error.
- Nefarious disjunction since the era of specialization.
- Man of Hellenism and of the Renaissance lost.
- Some *counter-examples* in Portugal: João de Freitas Branco (1922-89), Jorge de Sena (1919-78), Rómulo de Carvalho (1906-97) ...
- Sociology of mathematics — the real problem?
- Confluence requires change of attitude on both sides, with much work ahead.

Music well positioned to "bridge the gap" ...

- Experience in **Computer-aided Musicology** course at Minho.

Acknowledgements

I thank Álvaro Iriarte Sanromán (CEHUM) for inviting me to give this talk and for many interesting (coffee time) discussions and suggestions about this and other topics.

I also thank José J. Almeida (mate in the Computer-aided Musicology course) for his patience and attention whenever I bump into his office.