# Systems, behaviours and coinduction (II)

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- Moore and Mealy transducers
- Behavioural effects
- The general case: coalgebras
- Bisimulation and bisimilarity

# Moore transducers

| | |
|---|---|
| state space | $U$ |
| transition function | $\overline{\mathsf{nx}} : U^A \longleftarrow U$ |
| attribute (or label) | $\mathsf{at} : B \longleftarrow U$ |

i.e.,

$$p = \langle \overline{\mathsf{nx}}, \mathsf{at} \rangle : U^A \times B \longleftarrow U$$

Notation:

$$u \xrightarrow{a}_p u' \Leftrightarrow \overline{\mathsf{nx}}\, u\, a = u'$$

$$u \downarrow_p b \Leftrightarrow \mathsf{at}\, u = b$$

# Moore transducers

The behaviour of $p$ at (from) a state $u \in U$ is revealed by successive observations (experiments) triggered on input of different values $a \in A$:

$$[\![ p ]\!]\ u \quad = \quad [\mathsf{at}\ u, \mathsf{at}\ (\overline{\mathsf{nx}}\ u\ a_0), \mathsf{at}\ (\overline{\mathsf{nx}}\ (\overline{\mathsf{nx}}\ u\ a_0)\ a_1), \ldots]$$

$$\begin{aligned} [\![ p ]\!] u\ \underline{\mathsf{nil}} &= \mathsf{at}\ u \\ [\![ p ]\!] u\ (a : t) &= [\![ p ]\!]\ (\overline{\mathsf{nx}}\ u\ a)\ t \end{aligned}$$

which means that

> Moore behaviours are elements of $B^{A^*}$
> (depicted as rooted trees whose branches are labelled by sequences of inputs and leaves by $B$ values)

# Moore morphisms

A morphism

$$h : q \longleftarrow p$$

where

$$\begin{array}{rcl} p & = & \langle \overline{\mathsf{nx}}, \mathsf{at} \rangle : U^A \times B \longleftarrow U \\ q & = & \langle \overline{\mathsf{nx}}', \mathsf{at}' \rangle : V^A \times B \longleftarrow V \end{array}$$

is a function $h : V \longleftarrow U$ such that

$$\begin{array}{ccc} U & \xrightarrow{\;p\;} & U^A \times B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h^A \times \mathsf{id}} \\ V & \xrightarrow{\;q\;} & V^A \times B \end{array}$$

To avoid the explicit use of exponentials, the diagram can be decomposed into:

# Moore morphisms

$$\begin{CD} U @>{\mathsf{at}}>> B \\ @V{h}VV @VV{\mathsf{id}}V \\ V @>{\mathsf{at}'}>> B \end{CD}$$

and

$$\begin{CD} U \times A @>{nnx}>> U \\ @V{h \times \mathsf{id}}VV @VV{h}V \\ V \times A @>{\mathsf{nx}'}>> V \end{CD}$$

corresponding to

$$\begin{aligned} \mathsf{at}' \cdot h &= \mathsf{at} \\ \mathsf{nx}' \cdot (h \times \mathsf{id}) &= h \cdot \mathsf{nx} \end{aligned}$$

# Moore morphisms

Clearly, morphisms preserve attributes and transitions

$$
\begin{array}{ll}
 & u \xrightarrow{a}_p u' \quad \text{and} \quad u \downarrow_p b \\
\Leftrightarrow & \qquad \{ \text{ definition } \} \\
 & \mathsf{nx}(u, a) = u' \quad \text{and} \quad \mathsf{at}\, u = b \\
\Leftrightarrow & \qquad \{ \text{ Liebniz } \} \\
 & h\, \mathsf{nx}(u, a) = h\, u' \quad \text{and} \quad \mathsf{at}\, u = b \\
\Leftrightarrow & \qquad \{ \; h \text{ is a morphism } \} \\
 & \mathsf{nx}'(h\, u, a) = h\, u' \quad \text{and} \quad \mathsf{at}'\, h\, u = b \\
\Leftrightarrow & \qquad \{ \text{ definition } \} \\
 & h\, u \xrightarrow{a}_q h\, u' \quad \text{and} \quad h\, u \downarrow_q b
\end{array}
$$

# The final Moore transducer

Moore behaviours organise themselves into a final Moore machine over $B^{A^*}$

$$\omega = \langle \overline{\mathsf{nx}}_\omega, \mathsf{at}_\omega \rangle : (B^{A^*})^A \times B \longleftarrow B^{A^*}$$

where

$$\mathsf{at}_\omega\, f \quad = \quad f\, \mathsf{nil} \qquad \text{ie, the value before any input}$$

$$\overline{\mathsf{nx}}_\omega\, f\, a \quad = \quad \lambda\, s\, .\;\; f\,(a : s) \qquad \text{every input determines its evolution}$$

# The final Moore transducer

**Th:** Coalgebra $\omega$ is the final coalgebra for $\mathsf{T}\,X = X^A \times B$

because

**1.** For any $p = \langle \overline{\mathsf{nx}}, \mathsf{at} \rangle$, $[\![ p ]\!]$ is a Moore morphism $[\![ p ]\!] : \omega \longleftarrow p$

$$
\begin{array}{ll}
 & \mathsf{at}_\omega \cdot [\![ p ]\!] \;=\; \mathsf{at} \\
\Leftrightarrow & \quad \{ \text{ introduction of variables } \} \\
 & \mathsf{at}_\omega([\![ p ]\!]\, u) \;=\; \mathsf{at}\, u \\
\Leftrightarrow & \quad \{ \text{ definition of } \mathsf{at}_\omega \} \\
 & ([\![ p ]\!]\, u)\, \mathsf{nil} \;=\; \mathsf{at}\, u \\
\Leftrightarrow & \quad \{ \text{ definition of } [\![ p ]\!] \} \\
 & \textsc{True}
\end{array}
$$

# The final Moore transducer

$$\mathsf{nx}_\omega \cdot ([\![(p)]\!] \times \mathsf{id}) \;=\; [\![(p)]\!] \cdot \mathsf{nx}$$

$\Leftrightarrow$ { introduction of variables and application }

$$\mathsf{nx}_\omega([\![(p)]\!]\, u, a) \;=\; [\![(p)]\!]\, \mathsf{nx}\,(u, a)$$

$\Leftrightarrow$ { definition of $\mathsf{nx}_\omega$ }

$$\lambda\, s\, .\;\; ([\![(p)]\!]\, u)\,(a : s) \;=\; [\![(p)]\!]\, \mathsf{nx}\,(u, a)$$

$\Leftrightarrow$ { introduction of variables and application }

$$([\![(p)]\!]\, u)\,(a : t) \;=\; ([\![(p)]\!]\, \mathsf{nx}\,(u, a))\; t$$

$\Leftrightarrow$ { definition of $[\![(p)]\!]$ }

TRUE

# The final Moore transducer

2. … and is unique

Exercise. Prove uniqueness (by induction on $A^*$)

# Instances of Moore transducers

$$Queue = \langle \overline{\mathsf{nx}}, \mathsf{at} \rangle : (E^*)^{E+\mathbf{1}} \times ((E + \mathbf{1}) \times \mathbf{2}) \longleftarrow E^*$$

with

$$\begin{array}{rl} \mathsf{at} & = \langle \mathsf{top}, \mathsf{isempty?} \rangle \\ \text{where} \quad \mathsf{top}\ s & = (s = \underline{\mathsf{nil}} \rightarrow \iota_2 *, \iota_1(\mathsf{last}\, s)\ ) \\ \mathsf{isempty?}\ s & = s = \underline{\mathsf{nil}} \end{array}$$

$$\begin{array}{rl} \mathsf{nx} & = [\mathsf{enq}, \mathsf{deq}] \cdot \mathsf{dl} \\ \text{where} \quad \mathsf{enq}\ (s, e) & = e : s \\ \mathsf{deq}\ (s, *) & = (s = \underline{\mathsf{nil}} \rightarrow s, (\mathsf{blast}\, s)) \end{array}$$

# Instances of Moore transducers

Make $B = \mathbf{2}$ in $\mathsf{T}\,X = X^A \times B$.
The carrier (or state space) of the corresponding final coalgebra is

$$\mathbf{2}^{A^*} \;\cong\; \mathcal{P}A^*$$

and its dynamics is $\langle \overline{\mathsf{nx}}_\omega, \mathsf{at}_\omega \rangle : (\mathcal{P}A^*)^A \times \mathbf{2} \longleftarrow \mathcal{P}A^*$
where

$$\begin{array}{rcl} \mathsf{at}_\omega\, L & = & \underline{\mathsf{nil}} \in L \\ \overline{\mathsf{nx}}_\omega\, L & = & \lambda\, a\,.\ \{a \in A^* |\ (a : s) \in L\} \end{array}$$

Exercise. ... what are we talking about?
Exercise. Make $A = \mathbf{1}$ in $\mathsf{T}\,X = X^A \times B$. What comes up?

# Mealy transducers

| | |
|---|---|
| state space | $U$ |
| reactive transition function | $\overline{\mathsf{ac}} : (U \times B)^A \longleftarrow U$ |

Notation:

$$u \xrightarrow{a/b}_p u' \quad \Leftrightarrow \quad \overline{\mathsf{ac}}\, u\, a = (u', b)$$

# Mealy transducers

The behaviour of $p$ at a state $u \in U$ is revealed by successive observations (experiments) triggered on input of different values $a \in A$:

$$[\![ p ]\!]\ u \quad = \quad [\pi_2(\overline{\mathsf{ac}}\, u\, a_0), \pi_2(\overline{\mathsf{ac}}\, (\pi_2(\overline{\mathsf{ac}}\, u\, a_0))\, a_1, ...]$$

$$\begin{aligned} [\![ p ]\!] u\, [a] &= \pi_2(\mathsf{ac}\, u\, a) \\ [\![ p ]\!] u\, (a : t) &= [\![ p ]\!]\, (\pi_1(\mathsf{ac}\, u\, a))\, t \end{aligned}$$

which means that

| Mealy behaviours are elements of $B^{A^+}$ |
| --- |

# Mealy transducers

Mealy behaviours can alternatively be regarded as

causal functions from $A^\omega$ to $B^\omega$

A causal function $f$ over streams is such that, for all $s, t \in A^\omega$ and $n \in \mathbb{N}$,

$$\langle \forall\ k\ :\ k \leq n :\ s\,k\ =\ t\,k \rangle \Rightarrow (f\,s\,n\ =\ f\,t\,n)$$

i.e, the $n$-th element of $f\,s$ depends only on the first $n$ elements of input stream $s$

... upon which the final Mealy automata can be defined:

# The final Mealy transducer

Mealy behaviours organise themselves into a final Mealy automata over $\Gamma = \{f : B^\omega \longleftarrow A^\omega | \ f \text{ is causal}\}$

$$\overline{\omega} : (\Gamma \times B)^A \longleftarrow \Gamma$$

where

$$\overline{\omega}\, f\, a \quad = \quad \langle \lambda\, s\, .\, \mathrm{tl}\ f(a : s), \mathrm{hd}\, f(a : r) \rangle$$

which means that

- the next state acts as $f$ after $a$ has been seen
- the output $\mathrm{hd}\, f(a : r)$ depends only on $f$ and $a$; therefore, the tail $r$ of the input stream is irrelevant.

# Exercises

Exercise. Characterize Mealy morphisms. Draw the corresponding diagram and derive an equational definition.

Exercise. Prove that the Mealy transducer over $\Gamma$ defined above is final.

- Moore and Mealy transducers
- Behavioural effects
- The general case: coalgebras
- Bisimulation and bisimilarity

# Non-determinism

Further behavioural effects can be introduced in the basic machines discussed so far by 'sophisticating' the corresponding signature functor. For example,

- non-determinism is captured by the powerset functor $\mathcal{P}$

| | | |
|---|---|---|
| Automata | $\mathsf{T}X = B \times X$ | $\mathsf{T}X = \mathcal{P}(B \times X)$ |
| Moore transducer | $\mathsf{T}X = X^A \times B$ | $\mathsf{T}X = \mathcal{P}(X)^A \times B$ |
| Mealy transducer | $\mathsf{T}X = (X \times B)^A$ | $\mathsf{T}X = \mathcal{P}(X \times B)^A$ |

# Example: non-deterministic automata

Coalgebras

$$p : \mathcal{P}(B \times U) \longleftarrow U$$

as relations

$$P : B \times U \longleftarrow U$$

through the relational transpose

$$p = \Lambda P \Leftrightarrow P = \in \cdot p$$

Notation:

$$(b, x')\, P\, x \Leftrightarrow (b, x') \in p\, x \Leftrightarrow x'\, P_b\, x \Leftrightarrow x \xrightarrow{b}_p x'$$

# Example: non-deterministic automata

**Th**: A morphism between two non-deterministic automata $p$ and $q$ satisfies

$$(\mathsf{id} \times h) \cdot P \;=\; Q \cdot h \tag{1}$$

because

$$(\mathsf{id} \times h) \cdot P \;=\; Q \cdot h$$

$\Leftrightarrow$ { relational transpose is an isomorphism }

$$\Lambda((\mathsf{id} \times h) \cdot P) \;=\; \Lambda(Q \cdot h)$$

$\Leftrightarrow$ { $\Lambda(f \cdot R) = \mathcal{P}f \cdot \Lambda R$ and $\Lambda(R \cdot f) = \Lambda R \cdot f$ and definition }

$$\mathcal{P}(\mathsf{id} \times h) \cdot \Lambda(\in \cdot p) \;=\; \Lambda(\in \cdot q) \cdot h$$

$\Leftrightarrow$ { $\Lambda(R \cdot f) = \Lambda R \cdot f$ }

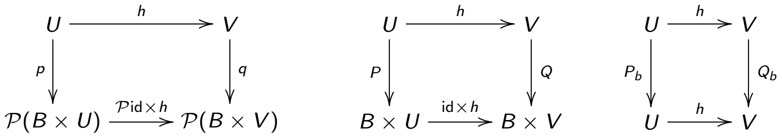
$$\mathcal{P}(\mathsf{id} \times h) \cdot \Lambda \in \cdot p \;=\; \Lambda \in \cdot q \cdot h$$

$\Leftrightarrow$ { $\Lambda \in = \mathsf{id}$ }

$$\mathcal{P}(\mathsf{id} \times h) \cdot p \;=\; q \cdot h$$

# Example: non-deterministic automata

Function $p$, relation $p$ and the $B$-indexed family of relations $\{P_b \mid b \in B\}$, all represent the same structure. Therefore, a morphism between non-deterministic automata can be defined by the commutativity of any of the following diagrams (of functions or relations, respectively):



# Example: non-deterministic automata

Therefore, equation (1) equivales to

$$\begin{array}{rcl} h \cdot P_b & \subseteq & Q_b \cdot h \\ Q_b \cdot h & \subseteq & h \cdot P_b \end{array}$$

entailing, respectively, preservation of $p$-transitions and reflection of $q$-transitions, i.e.,

$$\langle \forall\ u, u' \ :\ u, u' \in U :\ u \xrightarrow{b}_p u' \ \Rightarrow\ h\ u \xrightarrow{b}_q h\ u' \rangle \tag{2}$$

$$\langle \forall\ u, v \ :\ u \in U, v' \in V :\ h\ u \xrightarrow{b}_q v' \ \Rightarrow\ \langle \exists\ u' \ :\ u' \in U :\ u \xrightarrow{b}_p u' \ \wedge\ v' = h\ u' \rangle \rangle \tag{3}$$

because

# Example: non-deterministic automata

Proof of (2):

$$h \cdot P_b \subseteq Q_b \cdot h$$

$$\Leftrightarrow \quad \{ \text{ shunting } \}$$

$$P_b \subseteq h^\circ \cdot Q_b \cdot h$$

$$\Leftrightarrow \quad \{ \text{ PF transform } \}$$

$$\langle \forall\ u, u' \ :\ u, u' \in U :\ u'\, P_b\, u \Rightarrow u'(h^\circ \cdot Q_b \cdot h)u \rangle$$

$$\Leftrightarrow \quad \{ \text{ "guardanapo" rule } \}$$

$$\langle \forall\ u, u' \ :\ u, u' \in U :\ u'\, P_b\, u \Rightarrow (h\, u')Q_b(h\, u) \rangle$$

$$\Leftrightarrow \quad \{ P_b = (\xrightarrow{b}_p)^\circ \}$$

$$\langle \forall\ u, u' \ :\ u, u' \in U :\ u \xrightarrow{b}_p u' \Rightarrow h\, u \xrightarrow{b}_q h\, u' \rangle$$

# Example: non-deterministic automata

Proof of (2):

$$Q_b \cdot h \subseteq h \cdot P_b$$

$\Leftrightarrow$ { $\cdot R \vdash /R$ }

$$Q_b \subseteq (h \cdot P_b)/h$$

$\Leftrightarrow$ { definition of left division and PF transform }

$$\langle \forall\ v, v' \ : \ v, v' \in V : \ v' Q_b v \Rightarrow \langle \forall\ u \ : \ u \in U : \ v = h\,u \ \Rightarrow \ v'(h \cdot P_b)u \rangle \rangle$$

$\Leftrightarrow$ { quantifier trading (twice) }

$$\langle \forall\ v, v' \ : \ v, v' \in V \wedge v' Q_b v : \ \langle \forall\ u \ : \ u \in U \wedge v = h\,u : \ v'(h \cdot P_b)u \rangle \rangle$$

$\Leftrightarrow$ { quantifier nesting (twice, in opposite directions) }

$$\langle \forall\ u, v' \ : \ u \in U \wedge v' \in V : \ \langle \forall\ v \ : \ v \in V \wedge v = h\,u \wedge v' Q_b v : \ v'(h \cdot P_b)u \rangle \rangle$$

# Example: non-deterministic automata

$$\langle \forall\ u, v' \ :\ u \in U \wedge v' \in V :\ \langle \forall\ v\ :\ v \in V \wedge v = h\, u \wedge v' Q_b v :\ v'(h \cdot P_b) u \rangle \rangle$$

$\Leftrightarrow$ { quantifier trading }

$$\langle \forall\ u, v' \ :\ u \in U \wedge v' \in V :\ \langle \forall\ v\ :\ v = h\, u :\ (v \in V \wedge v' Q_b v) \Rightarrow v'(h \cdot P_b) u \rangle \rangle$$

$\Leftrightarrow$ { quantifier one-point rule }

$$\langle \forall\ u, v' \ :\ u \in U \wedge v' \in V :\ (h\, u \in V \wedge v' Q_b (h\, u)) \Rightarrow v'(h \cdot P_b) u \rangle$$

$\Leftrightarrow$ { $h$ type and definition of relational composition }

$$\langle \forall\ u, v' \ :\ u \in U \wedge v' \in V :\ v' Q_b v \Rightarrow \langle \exists\ u'\ :\ u' \in U :\ v' = h\, u \wedge u' P_b u \rangle \rangle$$

$\Leftrightarrow$ { $P_b = (\xrightarrow{b}_p)^\circ$ }

$$\langle \forall\ u, v' \ :\ u \in U \wedge v' \in V :\ v' Q_b v \Rightarrow \langle \exists\ u'\ :\ u' \in U :\ v' = h\, u \wedge u \xrightarrow{b}_p u' \rangle \rangle$$

# Partiality

| | | |
|---|---|---|
| Automata | $\mathsf{T}X = B \times X$ | $\mathsf{T}X = (B \times X) + \mathbf{1}$ |
| Moore transducer | $\mathsf{T}X = X^A \times B$ | $\mathsf{T}X = (X + \mathbf{1})^A \times B$ |
| Mealy transducer | $\mathsf{T}X = (X \times B)^A$ | $\mathsf{T}X = ((X \times B) + \mathbf{1})^A$ |

# In general: monads introduce behaviour

| | | |
|---|---|---|
| Automata | $\mathsf{T}X = B \times X$ | $\mathsf{T}X = \mathsf{B}(B \times X)$ |
| Moore transducer | $\mathsf{T}X = X^A \times B$ | $\mathsf{T}X = \mathsf{B}(X)^A \times B$ |
| Mealy transducer | $\mathsf{T}X = (X \times B)^A$ | $\mathsf{T}X = \mathsf{B}(X \times B)^A$ |

where B is a strong monad capturing a particular behavioural effect.

# Behaviour monads

- Partiality: $\mathsf{B} = \mathsf{Id} + \mathbf{1}$
- Non determinism: $\mathsf{B} = \mathcal{P}$
- Ordered non determinism: $\mathsf{B} = \mathsf{Id}^*$
- Monoidal labelling: $\mathsf{B} = \mathsf{Id} \times M$, with $M$ a monoid.
- 'Metric' non determinism: $\mathsf{B} = \mathsf{Bag}_M$ based on $\langle M, \oplus, \otimes \rangle$, where $\otimes$ distributes over $\oplus$, both defining Abelian monoids over $M$.

# Behaviour monads

$$\langle \mathsf{B}, \eta, \mu \rangle$$

where

$$\eta : \mathsf{Id} \Longleftarrow \mathsf{B} \quad \text{(to make a behavioural annotation)}$$
$$\mu : \mathsf{BB} \Longleftarrow \mathsf{B} \quad \text{(to flatten nested annotations)}$$

being strong entails the presence of right and left strength for context handling:

$$\mathsf{B}(\mathsf{Id} \times -) : \mathsf{B} \times - \Longleftarrow \mathsf{B} \times -$$
$$\mathsf{B}(- \times \mathsf{Id}) : - \times \mathsf{B} \Longleftarrow - \times \mathsf{B}$$

# Behaviour monads

Furthermore, Kleisli compositions

$$\delta r_{I,J} \;=\; \tau_{r_{I,J}} \bullet \tau_{l_{\mathrm{B}I,J}} \quad \text{and} \quad \delta l_{I,J} = \tau_{l_{I,J}} \bullet \tau_{r_{I,\mathrm{B}J}}$$

map

$$\mathrm{B}I \times \mathrm{B}J \;\text{ to }\; \mathrm{B}(I \times J)$$

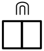
specifying a sort of sequential composition of B-computations

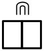
B is a commutative monad if $\delta r_{I,J} \;=\; \delta l_{I,J}$

... plus a handful of equational laws

- Moore and Mealy transducers
- Behavioural effects
- The general case: coalgebras
- Bisimulation and bisimilarity

# Algebras

a tool box: 

an assembly process: artifact $\xleftarrow{a}$  artifact

- algebras describe assembly processes
- and abstract data types as (initial) algebras (term algebras)
- emphasis is on construction

# Coalgebras

| | |
|---|---|
| a lens: | $\bigcirc\frown\bigcirc$ |
| an observation structure: | $\bigcirc\frown\bigcirc$ universe $\xleftarrow{c}$ universe |

- coalgebras describe observation structures (*i.e.*, transition systems)
- and abstract behaviour types as (final) coalgebras
- emphasis is on observation

# Typical lens

- ‘opaque’

$$\bigcirc\!\frown\!\bigcirc\; U \;=\; \mathbf{1}$$

- black & white

$$\bigcirc\!\frown\!\bigcirc\; U \;=\; \mathbf{2}$$

- colouring

$$\bigcirc\!\frown\!\bigcirc\; U \;=\; O$$

... in each case the colour set acts as a space classifier

# Typical lens

- partiality

$$\bigcirc\!\frown\!\bigcirc\; U \;=\; U + \mathbf{1}$$

- visible attributes

$$\bigcirc\!\frown\!\bigcirc\; U \;=\; O \times U$$

- external stimulus

$$\bigcirc\!\frown\!\bigcirc\; U \;=\; U^{I}$$

- non determinism

$$\bigcirc\!\frown\!\bigcirc\; U \;=\; \mathcal{P}U$$

# Question

Which lens shall we seek?

- The main criteria is to choose functors for which the final coalgebra does exist
- Such is the case of the all polynomial functors as well as finite powerset functor

# Coalgebras

A coalgebra for a functor T is any function from a set $U$ (its carrier) to $\mathsf{T}U$:

$$\alpha : \mathsf{T}U \longleftarrow U$$

For any functor T, if its space of behaviours can be made a T-coalgebra itself

$$\omega_{\mathsf{T}} : \mathsf{T}\nu_{\mathsf{T}} \longleftarrow \nu_{\mathsf{T}}$$

this is the final coalgebra: from any other T-coalgebra $p$ there is a unique morphism $[\![(p)]\!]$ making the following diagram to commute:

$$\begin{array}{ccc} \nu_{\mathsf{T}} & \xrightarrow{\omega_{\mathsf{T}}} & \mathsf{T}\nu_{\mathsf{T}} \\ {\scriptstyle [\![(p)]\!]}\uparrow & & \uparrow{\scriptstyle \mathsf{T}[\![(p)]\!]} \\ U & \xrightarrow{p} & \mathsf{T}U \end{array}$$

# Coalgebras

The universal property is equivalently captured by the following law:

$$k = [\![ p ]\!] \;\Leftrightarrow\; \omega_{\mathsf{T}} \cdot k = \mathsf{T}\ k \cdot p$$

- Existence $\Leftrightarrow$ definition principle (co-recursion)
- Uniqueness $\Leftrightarrow$ proof principle (co-induction)

From which:

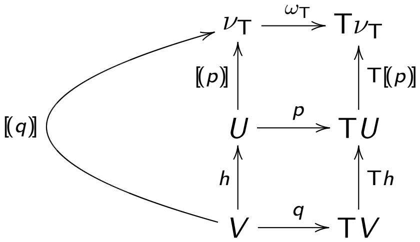
cancellation $\omega_{\mathsf{T}} \cdot [\![ p ]\!] = \mathsf{T}\ [\![ p ]\!] \cdot p$

reflection $[\![ \omega_{\mathsf{T}} ]\!] = \mathrm{id}_{\nu_{\mathsf{T}}}$

fusion $[\![ p ]\!] \cdot h = [\![ q ]\!]$ if $p \cdot h = \mathsf{T}\ h \cdot q$

# Coalgebras

Example: fusion law



# Coalgebras

Example: fusion law

$$\llbracket(p)\rrbracket \cdot h = \llbracket(q)\rrbracket$$

$\Leftrightarrow$ { universal law }

$$\omega \cdot \llbracket(p)\rrbracket \cdot h = \mathsf{T}(\llbracket(p)\rrbracket \cdot h) \cdot q$$

$\Leftrightarrow$ { cancellation law and $T$ functor }

$$\mathsf{T}\llbracket(p)\rrbracket \cdot p \cdot h = \mathsf{T}\llbracket(p)\rrbracket \cdot \mathsf{T}h \cdot q$$

$\Leftarrow$ { function equality }

$$p \cdot h = \mathsf{T}h \cdot q$$

# Coalgebras

From which one may generalise the fundamental result (proved above for the case of streams)

Th: morphisms preserve behaviour: $[\![(p)]\!] = [\![(q)]\!] \cdot h$

# Proof by coinduction

Example: $\mathsf{map}_f$ and generic laws

$$\mathsf{map}_{f \cdot g} \;=\; \mathsf{map}_f \cdot \mathsf{map}_g$$

defining $\mathsf{map}_f$ as follows:

$$\begin{array}{ccccc}
B^\omega & & \xrightarrow{\quad\quad\quad\omega_B\quad\quad\quad} & & B \times B^\omega \\
\uparrow{\scriptstyle \mathsf{map}_f} & & & & \uparrow{\scriptstyle \mathsf{id} \times \mathsf{map}_f} \\
A^\omega & \xrightarrow{\omega_A} & A \times A^\omega & \xrightarrow{f \times \mathsf{id}} & A \times B^\omega
\end{array}$$

# Proof by coinduction

$$\mathsf{map}_{f \cdot g} \;=\; \mathsf{map}_f \cdot \mathsf{map}_g$$

$\Leftrightarrow$ { map definition }

$$[\![((f \cdot g) \times \mathsf{id}) \cdot \omega]\!] \;=\; [\![(f \times \mathsf{id}) \cdot \omega]\!] \cdot \mathsf{map}_g$$

$\Leftarrow$ { coinduction fusion law }

$$(f \times \mathsf{id}) \cdot \omega \cdot \mathsf{map}_g \;=\; (\mathsf{id} \times \mathsf{map}_g) \cdot ((f \cdot g) \times \mathsf{id}) \cdot \omega$$

$\Leftrightarrow$ { coinduction cancellation law }

$$(f \times \mathsf{id}) \cdot (\mathsf{id} \times \mathsf{map}_g) \cdot (g \times \mathsf{id}) \cdot \omega \;=\; (\mathsf{id} \times \mathsf{map}_g) \cdot ((f \cdot g) \times \mathsf{id}) \cdot \omega$$

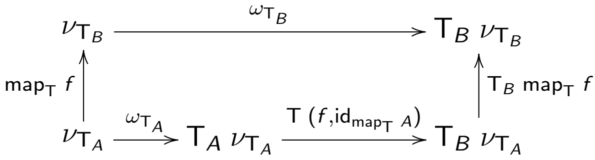
$\Leftrightarrow$ { functoriality }

$$((f \cdot g) \times \mathsf{map}_g) \cdot \omega \;=\; ((f \cdot g) \times \mathsf{map}_g) \cdot \omega$$

# Proof by coinduction

but this is just an instance of a more general result:

$$\mathsf{map}_{\mathsf{T}}\ (g \cdot f) \ = \ \mathsf{map}_{\mathsf{T}}\ g \cdot \mathsf{map}_{\mathsf{T}}\ f$$



# Proof by coinduction

In general one also gets:

$$\mathsf{map}_\mathsf{T}\,\mathsf{id}_A \;=\; \mathsf{id}_{\mathsf{map}_{\mathsf{T}_A}}$$
$$\mathsf{map}_\mathsf{T}\, f \cdot [\![(p)]\!]_\mathsf{T} \;=\; [\![(\mathsf{T}\,(f, \mathsf{id}) \cdot p)]\!]_\mathsf{T}$$

- function map extends to a functor mapping a set $A$ into the behaviour space of T-coalgebras parametric on $A$
- the last equation acts as an absorption law for coinductive extension

# Proof by coinduction

Example: Lambek's Lemma
The dynamics of the final coalgebra is an isomorphism

proof idea:

- Assume the existence of an inverse $\alpha_{\mathsf{T}}$ to $\omega_{\mathsf{T}} : \mathsf{T}\nu_{\mathsf{T}} \longleftarrow \nu_{\mathsf{T}}$.
  Then, $\alpha_{\mathsf{T}} \cdot \omega_{\mathsf{T}} = \mathsf{id}_{\nu_{\mathsf{T}}}$ and $\omega_{\mathsf{T}} \cdot \alpha_{\mathsf{T}} = \mathsf{id}_{\mathsf{T}\nu_{\mathsf{T}}}$
- Take one of this requirements and use it to conjecture a definition for $\alpha_{\mathsf{T}}$ (or an implementation ...)
  Note the use of the reflection law to introduce an anamorphism in the calculation, instead of eliminating one
- Then check the validity of this conjecture by verifying with it the other requirement

# Proof by coinduction

$$\alpha_{\mathsf{T}} \cdot \omega_{\mathsf{T}} = \mathsf{id}_{\nu_{\mathsf{T}}}$$

$\Leftrightarrow$ { reflection law }

$$\alpha_{\mathsf{T}} \cdot \omega_{\mathsf{T}} = [\![ \omega_{\mathsf{T}} ]\!]$$

$\Leftrightarrow$ { universal law }

$$\omega_{\mathsf{T}} \cdot \alpha_{\mathsf{T}} \cdot \omega_{\mathsf{T}} = \mathsf{T}(\alpha_{\mathsf{T}} \cdot \omega_{\mathsf{T}}) \cdot \omega_{\mathsf{T}}$$

$\Leftrightarrow$ { as a functor T preserves composition }

$$\omega_{\mathsf{T}} \cdot \alpha_{\mathsf{T}} \cdot \omega_{\mathsf{T}} = \mathsf{T}\alpha_{\mathsf{T}} \cdot \mathsf{T}\omega_{\mathsf{T}} \cdot \omega_{\mathsf{T}}$$

$\Leftrightarrow$ { cancel $\omega_{\mathsf{T}}$ from both sides & universal law }

$$\alpha_{\mathsf{T}} = [\![ \mathsf{T}\omega_{\mathsf{T}} ]\!]$$

# Proof by coinduction

$$
\begin{array}{cl}
 & \omega_{\mathsf{T}} \cdot \alpha_{\mathsf{T}} \\
= & \{ \text{ replace } \alpha_{\mathsf{T}} \text{ by the derived conjecture } \} \\
 & \omega_{\mathsf{T}} \cdot [\![ (\mathsf{T}\omega_{\mathsf{T}}) ]\!] \\
= & \{ \text{ } [\![ (\mathsf{T}\omega_{\mathsf{T}}) ]\!] \text{ is a morphism } \} \\
 & \mathsf{T}[\![ (\mathsf{T}\omega_{\mathsf{T}}) ]\!] \cdot \mathsf{T}\omega_{\mathsf{T}} \\
= & \{ \text{ as a functor T preserves composition } \} \\
 & \mathsf{T}([\![ (\mathsf{T}\omega_{\mathsf{T}}) ]\!] \cdot \omega_{\mathsf{T}}) \\
= & \{ \text{ just proved } \} \\
 & \mathsf{T}\,\mathsf{id}_{\nu_{\mathsf{T}}} \\
= & \{ \text{ as a functor T preserves identities } \} \\
 & \mathsf{id}_{(\mathsf{T}\mathsf{id}_{\nu_{\mathsf{T}}})}
\end{array}
$$

# Question

The powerset functor has not a final coalgebra. Why?

- Moore and Mealy transducers
- Behavioural effects
- The general case: coalgebras
- Bisimulation and bisimilarity

# Bisimulation

A bisimulation is a relation over the state spaces of two coalgebras, $p$ and $q$, which is closed for their dynamics, i.e.

$$(x, y) \in R \Rightarrow (p\,x, q\,y) \in \mathsf{T}R$$

which is PF-transformed to

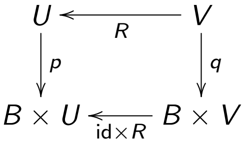
$$R \subseteq p^{\circ} \cdot (\mathsf{T}R) \cdot q$$

Shunting on $p^{\circ}$ yields

$$p \cdot R \subseteq (\mathsf{T}R) \cdot q$$

Note: signature functor $\mathsf{T}$ is now extended to a relator.

# Example: $\mathsf{T}\,X = B \times X$




$$p \cdot R \subseteq (\mathrm{id} \times R) \cdot q$$

$\Leftrightarrow$ { shunting }

$$R \subseteq p^\circ \cdot (\mathrm{id} \times R) \cdot q$$

$\Leftrightarrow$ { introducing variables }

$$\langle \forall\, u, v \;:\; u \in U, v \in V :\; u\, R\, v \;\Rightarrow\; u\,(p^\circ \cdot (\mathrm{id} \times R) \cdot q)\, v \rangle$$

$\Leftrightarrow$ { "guardanapo" rule }

$$\langle \forall\, u, v \;:\; u \in U, v \in V :\; u\, R\, v \;\Rightarrow\; p\, u\,(\mathrm{id} \times R)\, q\, v \rangle$$

$\Leftrightarrow$ { product }

$$\langle \forall\, u, v \;:\; u \in U, v \in V :\; u\, R\, v \;\Rightarrow\; \pi_1(p\, u) = \pi_1(q\, v) \;\wedge\; \pi_2(p\, u)\, R\, \pi_2(q\, v) \rangle$$

# Example: $\mathsf{T}\,X = \mathcal{P}X$

- Note that every powerset coalgebra can be regarded as the transpose of a binary relation through isomorphism

$$f = \Lambda R \quad \Leftrightarrow \quad R = \in \cdot f \tag{4}$$

- The powerset relator is defined by

$$\mathcal{P}R \;=\; (\in \backslash (R\cdot \in)) \cap (\in \backslash (R^{\circ}\cdot \in))^{\circ} \tag{5}$$

  where $\cap$ denotes relation intersection and $R \setminus S$ denotes relational division,

$$a(R \setminus S)c \;\Leftrightarrow\; \langle \forall\, b \;:\; b\, R\, a : \; b\, S\, c \rangle$$

  a relational operator whose semantics is captured by universal property

$$R \cdot X \subseteq S \quad \Leftrightarrow \quad X \subseteq R \setminus S \tag{6}$$

Then,

# Example: $\mathsf{T}\,X = \mathcal{P}X$

$$p \cdot R \subseteq (\mathcal{P}R) \cdot q$$

$\Leftrightarrow$ { let $p, q := \Lambda P, \Lambda Q$, unfold $\mathcal{P}R$ (5) }

$$(\Lambda P) \cdot R \subseteq (\in \backslash (R \cdot \in)) \cap (\in \backslash (R^\circ \cdot \in))^\circ \cdot (\Lambda Q)$$

$\Leftrightarrow$ { distribution (since $\Lambda Q$ is a function) }

$$(\Lambda P) \cdot R \subseteq (\in \backslash (R \cdot \in)) \cdot (\Lambda Q) \ \wedge \ (\Lambda P) \cdot R \subseteq (\in \backslash (R^\circ \cdot \in))^\circ \cdot (\Lambda Q)$$

$\Leftrightarrow$ { property $R \backslash (S \cdot f) = (R \backslash S) \cdot f$ ; converses }

$$(\Lambda P) \cdot R \subseteq \in \backslash (R \cdot \in \cdot \Lambda Q) \ \wedge \ R^\circ \cdot (\Lambda P)^\circ \subseteq (\Lambda Q)^\circ \cdot (\in \backslash (R^\circ \cdot \in))$$

$\Leftrightarrow$ { shunting and property above }

$$(\Lambda P) \cdot R \subseteq \in \backslash (R \cdot \in \cdot \Lambda Q) \ \wedge \ (\Lambda Q) \cdot R^\circ \subseteq \in \backslash (R^\circ \cdot \in \cdot \Lambda P)$$

$\Leftrightarrow$ { (6) twice }

$$\in \cdot (\Lambda P) \cdot R \subseteq R \cdot \in \cdot \Lambda Q \ \wedge \ \in \cdot (\Lambda Q) \cdot R^\circ \subseteq R^\circ \cdot \in \cdot \Lambda P$$

$\Leftrightarrow$ { cancellation $\in \cdot (\Lambda R) = R$ four times }

$$P \cdot R \subseteq R \cdot Q \ \wedge \ Q \cdot R^\circ \subseteq R^\circ \cdot P$$

# Example: $\mathsf{T}X = \mathcal{P}X$

The two conjuncts state that $R$ and its converse are simulations between state transition relations $P$ and $Q$, which corresponds to the Park-Milner definition:

- a bisimulation is a simulation such that its converse is also a simulation
- a simulation between relations $P$ and $Q$ is a relation $R$ such that, if $(p, q) \in R$, then for all $p'$ such that $(p', p) \in P$, then there is a $q'$ such that $(p', q') \in R$ and $(q', q) \in Q$

because

# Example: $\mathsf{T}\,X = \mathcal{P}X$

$$
\begin{array}{ll}
& P \cdot R \subseteq R \cdot Q \\
\Leftrightarrow & \quad \{\ S\cdot \vdash R\backslash\ \} \\
& R \subseteq P\backslash(R \cdot Q) \\
\Leftrightarrow & \quad \{\ \text{PF transform}\ \} \\
& \langle \forall\ u, v\ :\ u \in U \wedge v \in V :\ uRv \Rightarrow u(P\backslash(R \cdot Q))v \rangle \\
\Leftrightarrow & \quad \{\ \text{definition of right division}\ \} \\
& \langle \forall\ u, v\ :\ u \in U \wedge v \in V :\ uRv \Rightarrow \langle \forall\ u'\ :\ u' \in U \wedge u'Pu :\ u'(R \cdot Q)v \rangle \rangle \\
\Leftrightarrow & \quad \{\ \text{quantifier trading, nesting and trading again}\ \} \\
& \langle \forall\ u, u', v\ :\ u, u' \in U \wedge v \in V :\ uRv \wedge u'Pu \Rightarrow u'(R \cdot Q)v \rangle \\
\Leftrightarrow & \quad \{\ \text{relational composition}\ \} \\
& \langle \forall\ u, u', v\ :\ u, u' \in U \wedge v \in V :\ uRv \wedge u'Pu \Rightarrow \langle \exists\ v'\ :\ v' \in V :\ u'Rv' \wedge v'Qv \rangle \rangle
\end{array}
$$

# Example: $\mathsf{T}\,X = \mathcal{P}X$

and

$$
\begin{array}{ll}
& Q \cdot R^{\circ} \subseteq R^{\circ} \cdot P \\
\Leftrightarrow & \quad \{ \text{ by a similar argument } \} \\
& \langle \forall\ u, v, v' \ :\ u \in U \wedge v, v' \in V :\ uRv \wedge v' Q v \Rightarrow \langle \exists\ u' \ :\ u' \in U :\ u' R v' \wedge u' P u \rangle \rangle
\end{array}
$$

which jointly states that both $R$ and $R^{\circ}$ are simulations

# Example: $\mathsf{T}\,X = \mathcal{P}(B \times X)$

This result scales easily for $\mathsf{T}\,X = \mathcal{P}(B \times X)$ coalgebras, where it is usually expressed in terms of $B$-indexed families of transition relations:

- $R$ is a simulation between coalgebras $p$ and $q$ as before iff, for all $b \in B$, $u, u' \in U$ and $v \in V$,

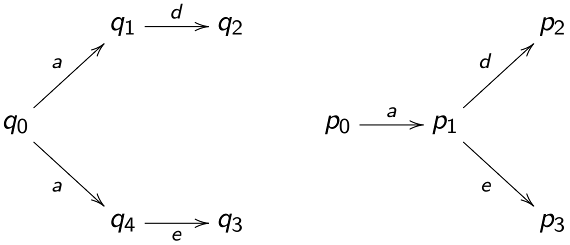
$$uRv \wedge u' \xrightarrow{b}_p u \Rightarrow \langle \exists\ v' \ :\ v' \in V :\ u' R v' \wedge v' \xrightarrow{b}_q v \rangle$$

- $R$ is a bisimulation iff both $R$ and $R^\circ$ are simulations

which leads to the usual definition of bisimulation in process algebra (cf, [Milner, 80])

# Example: $\mathrm{T}\,X = \mathcal{P}(B \times X)$

Example states $q_0$ and $p_0$ in coalgebras

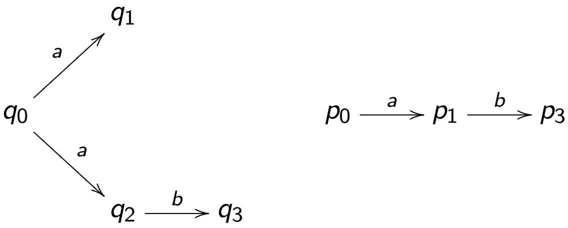



are related by simulation

$$\{\langle q_0, p_0\rangle, \langle q_1, p_1\rangle, \langle q_4, p_1\rangle, \langle q_2, p_2\rangle, \langle q_3, p_3\rangle\}$$

# Example: $\mathsf{T}\,X = \mathcal{P}(B \times X)$

Note, however, that, although there are simulations $R$ and $S$ containing pairs $(q_0, p_0)$ and $(p_0, q_0)$ in




the two states are not bisimilar.

Exercise. Compute relations $R$ and $S$ above and explain why $q_0$ and $p_0$ are not bisimilar.

Exercise. Compute the definition of bisimulation for the signature functor of a Moore and a Mealy transducer, respectively.

# Bisimulation as a Reynolds arrow

The definition of bisimulation brings to mind the "Reynolds arrow combinator"-pattern:

$$f(R \leftarrow S)g \quad \Leftrightarrow \quad f \cdot S \subseteq R \cdot g$$

leading to

$$R \text{ is a bisimulation} \quad \Leftrightarrow \quad p(\mathsf{T}R \leftarrow R)q \tag{7}$$

Note: Reynolds' arrow combinator is a relation on functions useful in expressing properties of functions, notably the "free theorem" of a polymorphic function $f$:

$$\mathsf{G}A \xleftarrow{\;f\;} \mathsf{T}A \text{ polymorphic} \quad \Leftrightarrow \quad \langle \forall\ R\ ::\ f(\mathsf{G}R \leftarrow \mathsf{T}R)f \rangle$$

# Reynolds-arrow laws

$$id \leftarrow id = id \tag{8}$$

$$(R \leftarrow S)^\circ = R^\circ \leftarrow S^\circ \tag{9}$$

$$(R \leftarrow V) \cdot (S \leftarrow U) \subseteq (R \cdot S) \leftarrow (V \cdot U) \tag{10}$$

$$R \leftarrow S \subseteq V \leftarrow U \Leftarrow R \subseteq V \wedge U \subseteq S \tag{11}$$

$$k(f \leftarrow g)h \Leftrightarrow k \cdot g = f \cdot h \tag{12}$$

$$(f \leftarrow g^\circ)h = f \cdot h \cdot g \tag{13}$$

# Reynolds-arrow laws

Property (11) entails monotonicity on the left hand side, thus,

$$S \leftarrow R \subseteq (S \cup V) \leftarrow R \tag{14}$$

$$\top \leftarrow S = \top \tag{15}$$

and anti-monotonicity on the right hand side:

$$R \leftarrow \bot = \top \tag{16}$$

as well as two distributive properties:

$$S \leftarrow (R_1 \cup R_2) = (S \leftarrow R_1) \cap (S \leftarrow R_2) \tag{17}$$

$$(S_1 \cap S_2) \leftarrow R = (S_1 \leftarrow R) \cap (S_2 \leftarrow R) \tag{18}$$

# Bisimulation: Properties

- The converse of a bisimulation is also a bisimulation

$$
\begin{array}{ll}
 & R \text{ is a bisimulation} \\
\Leftrightarrow & \quad \{ \ (7) \ \} \\
 & p(\mathsf{T}R \leftarrow R)q \\
\Leftrightarrow & \quad \{ \ \text{converse} \ \} \\
 & q(\mathsf{T}R \leftarrow R)^{\circ} p \\
\Leftrightarrow & \quad \{ \ (9) \ \} \\
 & p((\mathsf{T}R)^{\circ} \leftarrow R^{\circ})q \\
\Leftrightarrow & \quad \{ \ \text{relator } \mathsf{T} \ \} \\
 & q(\mathsf{T}(R^{\circ}) \leftarrow R^{\circ})p \\
\Leftrightarrow & \quad \{ \ (7) \ \} \\
 & R^{\circ} \text{ is a bisimulation}
\end{array}
$$

- Composition of bisimulations is a bisimulation
  by property (10), as can be checked by parsing its pointwise version: for all suitably typed coalgebras $p$ and $q$,

$$\langle \exists\, z \ :: \ p(\mathsf{T}S \leftarrow S)z \wedge z(\mathsf{T}R \leftarrow R)q \rangle \quad \Rightarrow \quad p(\mathsf{T}(S \cdot R) \leftarrow (S \cdot R))q$$

# Bisimulation: Properties

- the identity relation *id* is a bisimulation

$$
\begin{array}{ll}
 & p(\mathsf{T}\,id \leftarrow id)q \\
\Leftrightarrow & \quad \{ \text{ relator T } \} \\
 & p(id \leftarrow id)q \\
\Leftrightarrow & \quad \{ \text{ (8) } \} \\
 & p = q
\end{array}
$$

- the empty relation $\bot$ is a bisimulation

$$
\begin{array}{ll}
 & \langle \forall\ p, q \ :: \ p(\mathsf{T}\bot \leftarrow \bot)q \rangle \\
\Leftrightarrow & \quad \{ \text{ PF-transform } \} \\
 & \langle \forall\ p, q \ :: \ p(\mathsf{T}\bot \leftarrow \bot)q \Leftrightarrow \text{TRUE} \rangle \\
\Leftrightarrow & \quad \{ \text{ PF-transform } \} \\
 & \mathsf{T}\bot \leftarrow \bot = \top \\
\Leftrightarrow & \quad \{ \text{ (16) } \} \\
 & \text{TRUE}
\end{array}
$$

# Bisimulation: Properties

- bisimulations are closed under union

$$p(\mathsf{T}R_1 \leftarrow R_1)q \wedge p(\mathsf{T}R_2 \leftarrow R_2)q \quad \Rightarrow \quad p(\mathsf{T}(R_1 \cup R_2) \leftarrow (R_1 \cup R_2))q \qquad (19)$$

stems from properties (11,14) and (17). First we PF-transform (19) to

$$(\mathsf{T}R_1 \leftarrow R_1) \cap (\mathsf{T}R_2 \leftarrow R_2) \quad \subseteq \quad \mathsf{T}(R_1 \cup R_2) \leftarrow (R_1 \cup R_2)$$

and reason:

$$\begin{array}{cl}
& (\mathsf{T}R_1 \leftarrow R_1) \cap (\mathsf{T}R_2 \leftarrow R_2) \\
\subseteq & \quad \{ \ (14)\ (\text{twice})\ ;\ \text{monotonicity of } \cap \ \} \\
& ((\mathsf{T}R_1 \cup \mathsf{T}R_2) \leftarrow R_1) \cap ((\mathsf{T}R_1 \cup \mathsf{T}R_2) \leftarrow R_2) \\
= & \quad \{ \ (17) \ \} \\
& (\mathsf{T}R_1 \cup \mathsf{T}R_2) \leftarrow (R_1 \cup R_2) \\
\subseteq & \quad \{ \ \mathsf{F} \text{ is monotonic; } (11) \ \} \\
& \mathsf{T}(R_1 \cup R_2) \leftarrow (R_1 \cup R_2)
\end{array}$$

# Bisimulation: Properties

- any coalgebra morphism is a bisimulation (why?)
- behavioural equivalence is a bisimulation.

$$
\begin{array}{ll}
 & p(\mathsf{T}([\![p]\!]^\circ \cdot [\![q]\!]) \leftarrow [\![p]\!]^\circ \cdot [\![q]\!])q \\
\Leftrightarrow & \quad \{\ \text{definition}\ \} \\
 & [\![p]\!]^\circ \cdot [\![q]\!] \subseteq p^\circ \cdot \mathsf{T}([\![p]\!]^\circ \cdot [\![q]\!]) \cdot q \\
\Leftrightarrow & \quad \{\ \text{relators}\ \} \\
 & [\![p]\!]^\circ \cdot [\![q]\!] \subseteq p^\circ \cdot \mathsf{T}[\![p]\!]^\circ \cdot \mathsf{T}[\![q]\!] \cdot q \\
\Leftrightarrow & \quad \{\ \text{converse}\ \} \\
 & [\![p]\!]^\circ \cdot [\![q]\!] \subseteq (\mathsf{T}[\![p]\!] \cdot p)^\circ \cdot \mathsf{T}[\![q]\!] \cdot q \\
\Leftrightarrow & \quad \{\ \text{universal property of coinductive extension}\ \} \\
 & [\![p]\!]^\circ \cdot [\![q]\!] \subseteq (\omega \cdot [\![p]\!])^\circ \cdot \omega \cdot [\![q]\!] \\
\Leftrightarrow & \quad \{\ \text{converse}\ \} \\
 & [\![p]\!]^\circ \cdot [\![q]\!] \subseteq [\![p]\!]^\circ \cdot \omega^\circ \cdot \omega \cdot [\![q]\!] \\
\Leftrightarrow & \quad \{\ \text{Lambek (final coalgebra is an isomorphism)}\ \} \\
 & \textsc{True}
\end{array}
$$

# Bisimilarity

**Def.** Two states, $u$ and $v$, from the same or different coalgebras, are bisimilar iff they are related by a bisimulation, i.e.,

$$u \sim v \;\Leftrightarrow\; \langle \exists\, R \;:\; R \subseteq U \times V :\; uRv \wedge R \text{ is a bisimulation} \rangle$$

**Th.** Bisimilarity is an equivalence relation.

**Th.** The set of all bisimulations, defined between two coalgebras, over state spaces $U$ and $V$, is a complete lattice, ordered by $\subseteq$, whose top is the restriction of $\sim$ to $U \times V$.

**Exercise.** Prove both theorems.