Foundations of the PF relational calculus

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Recalling...

Monotonicity:

All operations are monotonic, eg.

$$\frac{R \subseteq S \quad T \subseteq U}{(R \cdot T) \subseteq (S \cdot U)} \qquad \frac{R \subseteq S}{R^\circ \subseteq S^\circ}$$

Composition:

- Composition is associative: $R \cdot (S \cdot T) = (R \cdot S) \cdot T$
- Identity: $R \cdot id = id \cdot R = R$
- Empty relation: $R \cdot \bot = \bot \cdot R = \bot$

Recalling...

Pointfree Relational Equality:

- **Cyclic inclusion** ("ping-pong") rule:

$$R = S \quad \equiv \quad R \subseteq S \wedge S \subseteq R$$

- **Indirect equality** rules:

$$\begin{aligned} R = S \quad & \equiv \quad \langle \forall\, X \;::\; (X \subseteq R \equiv X \subseteq S) \rangle \\ & \equiv \quad \langle \forall\, X \;::\; (R \subseteq X \equiv S \subseteq X) \rangle \end{aligned}$$

Relational algebra: converse

Properties:

$$\circ\text{-}universal: \quad X^\circ \subseteq Y \equiv X \subseteq Y^\circ \tag{1}$$

$$\circ\text{-}monotonicity: \quad R \subseteq S \equiv R^\circ \subseteq S^\circ \tag{2}$$

Then:

$$Involution: \quad (R^\circ)^\circ = R \tag{3}$$

$$Contravariance: \quad (R \cdot S)^\circ = S^\circ \cdot R^\circ \tag{4}$$

These can be proved from $^\circ$-**universal** by (elegant) indirect proofs (cf. exercises later on):

Relation algebra: meet and join

Properties:

$\cap$-*universal:* $$X \subseteq (R \cap S) \quad \equiv \quad (X \subseteq R) \wedge (X \subseteq S) \tag{5}$$

$\cup$-*universal:* $$R \cup S \subseteq X \quad \equiv \quad (R \subseteq X) \wedge (S \subseteq X) \tag{6}$$

Then

- Converse distributes over $\cap$:

$$(R \cap S)^\circ = R^\circ \cap S^\circ \tag{7}$$

- Converse distributes over $\cup$:

$$(R \cup S)^\circ = R^\circ \cup S^\circ \tag{8}$$

(sample calculational proof follows)

Exercise 1: Complete the following calculation by indirect equality:

$$
\begin{array}{ll}
 & X \subseteq R^\circ \cap S^\circ \\
\equiv & \qquad \{ \ \ldots \ \} \\
 & (X \subseteq R^\circ) \wedge (X \subseteq S^\circ) \\
\equiv & \qquad \{ \ \ldots \ \} \\
 & (X^\circ \subseteq R) \wedge (X^\circ \subseteq S) \\
\equiv & \qquad \{ \ \ldots \ \} \\
 & X^\circ \subseteq R \cap S \\
\equiv & \qquad \{ \ \ldots \ \} \\
 & X \subseteq (R \cap S)^\circ \\
:: & \qquad \{ \text{ indirection } \} \\
 & R^\circ \cap S^\circ = (R \cap S)^\circ
\end{array}
$$

Relational calculus: functions

Shunting rules:

$$f \cdot R \subseteq S \quad \equiv \quad R \subseteq f^\circ \cdot S \tag{9}$$

$$R \cdot f^\circ \subseteq S \quad \equiv \quad R \subseteq S \cdot f \tag{10}$$

Equality:

$$f \subseteq g \equiv f = g \equiv f \supseteq g \tag{11}$$

"Cyclic inclusion" calculation of the equality rule (11) follows.

Proof of functional equality

$$
\begin{aligned}
& f \subseteq g \\
\equiv \quad & \quad \{ \text{ identity } \} \\
& f \cdot id \subseteq g \\
\equiv \quad & \quad \{ \text{ shunting on } f \ \} \\
& id \subseteq f^{\circ} \cdot g \\
\equiv \quad & \quad \{ \text{ shunting on } g \ \} \\
& id \cdot g^{\circ} \subseteq f^{\circ} \\
\equiv \quad & \quad \{ \text{ converses; identity } \} \\
& g \subseteq f
\end{aligned}
$$

Adding structure to the calculus

Note a recurrent **pattern** in several laws above:

$$\underbrace{X^{\circ}}_{f\ X} \subseteq Y \quad \equiv \quad X \subseteq \underbrace{Y^{\circ}}_{g\ Y}$$

$$\underbrace{(h\cdot)X}_{f\ X} \subseteq Y \quad \equiv \quad X \subseteq \underbrace{(h^{\circ}\cdot)Y}_{g\ Y}$$

$$\underbrace{X(\cdot h^{\circ})}_{f\ X} \subseteq Y \quad \equiv \quad X \subseteq \underbrace{Y(\cdot h)}_{g\ Y}$$

as well as in

$$\underbrace{(d\times)q}_{f\ q} \leq n \quad \equiv \quad q \leq \underbrace{n(/d)}_{g\ n}$$

where $(/d)$ denotes integral division (in $\mathbb{N}_0$).

Back to the primary school desk

The **integral division** algorithm

$$\begin{array}{r|l} 7 & 2 \\ \hline 1 & 3 \end{array} \qquad 2 \times 3 + 1 = 7 \quad , \text{ “ie.” } \qquad 3 = 7/2$$

However

$$\begin{array}{r|l} 7 & 2 \\ \hline 3 & 2 \end{array} \qquad 2 \times 2 + 3 = 7 \quad \wedge \quad 2 \neq 7/2$$

$$\begin{array}{r|l} 7 & 2 \\ \hline 5 & 1 \end{array} \qquad 2 \times 1 + 5 = 7 \quad \wedge \quad 1 \neq 7/2$$

In fact:

$$\begin{array}{r|l} n & d \\ \hline r & q \end{array} \qquad q = n/d \;\equiv\; d \times q + r = n$$

provided q is the largest such q (r smallest)

Back to the primary school desk

So:

- Quotient is a supremum:

$$\begin{aligned} n/d &= \langle \bigvee\, q \;::\; \langle \exists\, r \;::\; d \times q + r = n \rangle \rangle \\ &= \langle \bigvee\, q \;::\; d \times q \leq n \rangle \end{aligned}$$

- Maths teachers tell: *it takes a while before children master the "$\bigvee$ semantics"!*
- What about you? Can you easily reason about n/d in this format?
- Challenge: Try and prove $(n/m)/d = n/(d \times m)$.

Proposed alternative: **al-djabr** rule

$$q \times d \leq n \;\equiv\; q \leq n/d \qquad (12)$$

"universal" property of integral division

“Al-djabr” calculation instead

$$
\begin{array}{ll}
 & q \leq (n/m)/d \\
\equiv & \quad \{ \text{ “al-djabr” (12) } \} \\
 & q \times d \leq n/m \\
\equiv & \quad \{ \text{ “al-djabr” (12) } \} \\
 & (q \times d) \times m \leq n \\
\equiv & \quad \{ \times \text{ is associative } \} \\
 & q \times (d \times m) \leq n \\
\equiv & \quad \{ \text{ “al-djabr” (12) } \} \\
 & q \leq n/(d \times m) \\
:: & \quad \{ \text{ indirection } \} \\
 & (n/m)/d = n/(d \times m)
\end{array}
$$

(Generic) indirect equality

Note the use of **indirect equality** rule

$$(q \leq x \equiv q \leq y) \equiv (x = y)$$

in fact valid for $\leq$ **any** partial order.

Exercise 2: Derive from (12) the two *cancellation* laws

$$q \leq (q \times d)/d \qquad (13)$$

$$(n/d) \times d \leq n \qquad (14)$$

and *reflexion* law:

$$n/d \geq 1 \equiv d \leq n \qquad (15)$$

□

Galois connections

n/d is an example of operation involved in a **Galois** connection:

$$\underbrace{q \times d}_{f\ q} \leq n \quad \equiv \quad q \leq \underbrace{n/d}_{g\ n}$$

In general, for **preorders** $(A, \leq)$ and $(B, \sqsubseteq)$ and

$$(A, \leq) \underset{f}{\overset{g}{\rightleftarrows}} (B, \sqsubseteq) \tag{16}$$

(f, g) are *Galois connected* iff. . .

Galois adjoints

$$\underbrace{f}_{\text{lower adjoint}} \; b \leq a \quad \equiv \quad b \sqsubseteq \underbrace{g}_{\text{upper adjoint}} \; a$$

that is

$$f^{\circ} \cdot \leq \quad = \quad \sqsubseteq \cdot g$$

Remarks:

- Galois (connected) adjoints enjoy a number of interesting **generic** properties
- *Very elegant* — **calculational** — way of performing *equational* reasoning (including *logical* deduction)

Basic properties

Cancellation:

$$(f \cdot g)a \leq a \quad \text{and} \quad b \sqsubseteq (g \cdot f)b$$

Distribution (in case of lattice structures):

$$\begin{array}{rcl} f(a \sqcup a') & = & (f\ a) \vee (f\ a') \\ g(b \wedge b') & = & (g\ b) \sqcap (g\ b') \end{array}$$

Conversely,

- If f distributes over $\sqcup$ then it has an upper adjoint g ($f^{\#}$)
- If g distributes over $\wedge$ then it has a lower adjoint f ($g^{\flat}$)

Other properties

If (f, g) are Galois connected,

- f (g) **uniquely** determines g (f) — thus the $\_^\flat$, $\_^\sharp$ notations
- f and g are **monotonic**
- (g, f) are also Galois connected — just **reverse** the orderings
- $f = f \cdot g \cdot f$ and $g = g \cdot f \cdot g$

etc

Summary

$(f\ b) \leq a \equiv b \sqsubseteq (g\ a)$		
Description	$f = g^\flat$	$g = f^\sharp$
Definition	$f\ b = \bigwedge\{a : b \sqsubseteq g\ a\}$	$g\ a = \bigsqcup\{b : f\ b \leq a\}$
Cancellation	$f(g\ a) \leq a$	$b \sqsubseteq g(f\ b)$
Distribution	$f(b \sqcup b') = (f\ b) \vee (f\ b')$	$g(a' \sqcap a) = (g\ a') \sqcap (g\ a)$
Monotonicity	$b \sqsubseteq b' \Rightarrow f\ b \leq f\ b'$	$a \leq a' \Rightarrow g\ a \sqsubseteq g\ a'$

In the sequel we will re-interpret the relational operators we've seen so far as Galois adjoints.

Converse

$(f\ X) \subseteq Y \equiv X \subseteq (g\ Y)$			
Description	$f = g^\flat$	$g = f^\sharp$	**Obs.**
converse	$(\_)^\circ$	$(\_)^\circ$	$bR^\circ a \equiv aRb$

Thus:

Cancellation $(R^\circ)^\circ = R$

Monotonicity $R \subseteq S \equiv R^\circ \subseteq S^\circ$

Distributions $(R \cap S)^\circ = R^\circ \cap S^\circ, (R \cup S)^\circ = R^\circ \cup S^\circ$

Functions

$(f\ X) \subseteq Y \equiv X \subseteq (g\ Y)$			
Description	$f = g^\flat$	$g = f^\sharp$	**Obs.**
shunting rule	$(h\cdot)$	$(h^\circ\cdot)$	NB: h is a function
"converse" shunting rule	$(\cdot h^\circ)$	$(\cdot h)$	NB: h is a function

Consequences:

Functional equality: $h \subseteq g \equiv \quad h = k \quad \equiv h \supseteq k$

Functional division: $h^\circ \cdot R = h \setminus R$

Question: what does $h \setminus R$ mean?

Relational division

$(f\ X) \subseteq Y \equiv X \subseteq (g\ Y)$			
Description	$f = g^\flat$	$g = f^\sharp$	**Obs.**
left-division	$(R\cdot)$	$(R \setminus\ )$	left-factor
right-division	$(\cdot R)$	$(\ / R)$	right-factor

that is,

$$R \cdot X \subseteq Y \equiv X \subseteq R \setminus Y \tag{17}$$

$$X \cdot R \subseteq Y \equiv X \subseteq Y / R \tag{18}$$

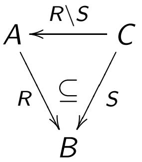
Immediate: $(R\cdot)$ and $(\cdot R)$ distribute over union:

$$\begin{array}{rcl} R \cdot (S \cup T) & = & (R \cdot S) \cup (R \cdot T) \\ (S \cup T) \cdot R & = & (S \cdot R) \cup (T \cdot R) \end{array}$$

Some intuition about relational division operators follows.

Relational (left) division

Left division abstracts a (pointwise) universal quantification



$$a(R \setminus S)c \;\equiv\; \langle \forall\, b \;:\; b\, R\, a :\; b\, S\, c \rangle \qquad (19)$$

Example:

$b\ R\ a$ = flight b carries passenger a

$b\ S\ c$ = flight b belongs to air-company c

$a\ (R \setminus S)\ c$ = passenger a is faithful to company c, that is, (s)he only flies company c.

Relational (right) division

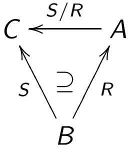
By taking converses we arrive at $S \,/\, R = (R^\circ \setminus S^\circ)^\circ$:

$$
\begin{array}{ll}
 & X \subseteq S \,/\, R \\
\equiv & \qquad \{ \text{ Galois connection } ((\cdot R),(/R)) \text{ } \} \\
 & X \cdot R \subseteq S \\
\equiv & \qquad \{ \text{ converses } \} \\
 & R^\circ \cdot X^\circ \subseteq S^\circ \\
\equiv & \qquad \{ \text{ Galois connection } ((R \cdot),(R \setminus)) \text{ } \} \\
 & X^\circ \subseteq R^\circ \setminus S^\circ \\
\equiv & \qquad \{ \text{ converses } \} \\
 & X \subseteq (R^\circ \setminus S^\circ)^\circ \\
:: & \qquad \{ \text{ indirection } \} \\
 & S \,/\, R = (R^\circ \setminus S^\circ)^\circ
\end{array}
$$

Relational (right) division

Therefore:

$$
\begin{array}{cl}
 & c(S \,/\, R)a \\
\equiv & \quad \{ \text{ above } \} \\
 & a(R^\circ \setminus S^\circ)c \\
\equiv & \quad \{ \text{ (19) } \} \\
 & \langle \forall\ b \ :\ b\ R^\circ a :\ b\ S^\circ c \rangle \\
\equiv & \quad \{ \text{ converses } \} \\
 & \langle \forall\ b \ :\ a\ R\ b :\ c\ S\ b \rangle
\end{array}
$$



Domain and range

$(f\ X) \subseteq Y \equiv X \subseteq (g\ Y)$			
Description	$f = g^{\flat}$	$g = f^{\sharp}$	**Obs.**
domain	δ	$(\top \cdot)$	lower $\subseteq$ restricted to coreflexives
range	ρ	$(\cdot \top)$	lower $\subseteq$ restricted to coreflexives

Thus

$$\delta\, R \subseteq \Phi \quad \equiv \quad R \subseteq R \cdot \Phi \tag{20}$$

$$\rho\, R \subseteq \Phi \quad \equiv \quad R \subseteq \Phi \cdot \top \tag{21}$$

etc.

Domain and split

The following fact holds:

$$\langle R, S\rangle^\circ \cdot \langle X, Y\rangle \quad = \quad (R^\circ \cdot X) \cap (S^\circ \cdot Y)$$

Corollary:

$$\delta\, R \quad = \quad \ker \langle id, R\rangle$$

Another consequence of the fact above:

$$\ker R \subseteq \ker (S \cdot R) \quad \Leftarrow \quad S \text{ entire}$$

Corollary:

$$\ker R \quad \subseteq \quad \ker (f \cdot R)$$

Modular law

Dedekind's rule, also known as the **modular law**:

$$R \cdot S \cap T \quad \subseteq \quad R \cdot (S \cap R^{\circ} \cdot T)$$

cf. analogy with $ab + c \quad \leq \quad a(b + a^{-1}c)$. Dually (apply converses and rename):

$$(R \cdot S) \cap T \quad \subseteq \quad (R \cap (T \cdot S^{\circ})) \cdot S$$

Symmetrical equivalent statement:

$$(R \cdot S) \cap T \quad \subseteq \quad (R \cap (T \cdot S^{\circ})) \cdot (S \cap (R^{\circ} \cdot T))$$

$=$ “weak right-distribution of meet over composition”.