Proof obligation discharge using the PF transform

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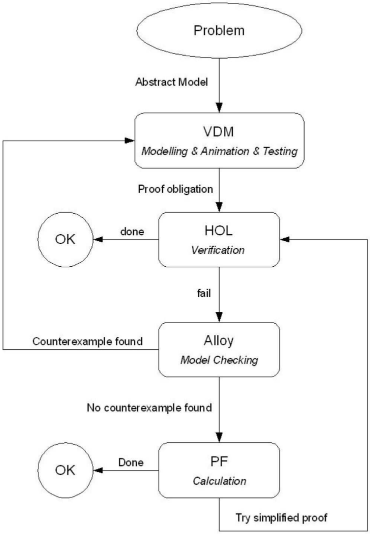
Summary

Learning outcomes:

- Discharging proof obligations via PF-transform. Pre/post conditions. Invariants.
- Extended static checking in the PF-style. PF-calculation of weakest pre-conditions for invariant maintenance.
- Three examples

Broad picture: a "all-in-one" strategy for PO discharge



Poof obligations in the PF-style

In general:

Input/output property preservation (functions)

Proof obligation

$$\langle \forall\, x \;:\; p\, x :\; q\,(f\; x)\rangle \tag{1}$$

stating that function f **ensures** *property* q *on its* **output** *every time property* p *holds on its* **input** *PF-transforms to*

$$f \cdot \Phi_p \;\subseteq\; \Phi_q \cdot f \quad \textit{cf. diagram} \quad \begin{tikzcd} A \arrow[d, "f"'] & A \arrow[l, "\Phi_p"'] \arrow[d, "f"] \\ B & B \arrow[l, "\Phi_q"] \end{tikzcd} \tag{2}$$

Predicates as "types"

We will write "type declaration"

$$\Phi_q \xleftarrow{f} \Phi_p \tag{3}$$

to mean (2).

Exercise 1: Show that (2) and

$$f \cdot \Phi_p \;\subseteq\; \Phi_q \cdot \top \tag{4}$$

are the same.

$\square$

Exercise 2: Prove the equivalence

$$\Phi_q \xleftarrow{id} \Phi_p \quad \equiv \quad q \Leftarrow p \tag{5}$$

$\square$

Exercises

Exercise 3: Infer from (3) and properties (59) to (61) the following ESC (*extended static checking*) properties:

$$\Phi_q \xleftarrow{f} \Phi_{p_1} \cup \Phi_{p_2} \quad \equiv \quad \Phi_q \xleftarrow{f} \Phi_{p_1} \ \wedge \ \Phi_q \xleftarrow{f} \Phi_{p_2} \tag{6}$$

$$\Phi_{q_1} \cdot \Phi_{q_2} \xleftarrow{f} \Phi_p \quad \equiv \quad \Phi_{q_1} \xleftarrow{f} \Phi_p \ \wedge \ \Phi_{q_2} \xleftarrow{f} \Phi_p \tag{7}$$

□

Exercise 4: Using (4) and the relational version of McCarthy's conditional combinator which follows,

$$c \rightarrow f, g \ = \ f \cdot \Phi_c \cup g \cdot \Phi_{\neg c} \tag{8}$$

infer the *conditional ESC* rule which follows:

$$\Phi_q \xleftarrow{c \rightarrow f, g} \Phi_p \quad \equiv \quad \Phi_q \xleftarrow{f} \Phi_p \cdot \Phi_c \ \wedge \ \Phi_q \xleftarrow{g} \Phi_p \cdot \Phi_{\neg c} \tag{9}$$

□

Relationship with Hoare Logic

Let us show that **Hoare triples** such as

$$\{p\}P\{q\} \tag{10}$$

are also instances of ESC proof obligations. First we spell out the meaning of (10):

$$\langle \forall\, s \;:\; p\ s : \; \langle \forall\, s' \;:\; s \xrightarrow{P} s' \;:\; q\ s' \rangle \rangle \tag{11}$$

Then (recording the meaning of program P as relation $[\![P]\!]$ on program states) we PF-transform (11) into

$$\Phi_p \subseteq [\![P]\!] \setminus (\Phi_q \cdot \top) \tag{12}$$

thanks to the introduction of relational (left) **division**,

$$b\ (R \setminus S)\ a \;\equiv\; \langle \forall\, c \;:\; c\ R\ b : \; c\ S\ a \rangle \tag{13}$$

Relationship with Hoare Logic

Thanks to "al-djabr" rule

$$R \cdot X \subseteq S \quad \equiv \quad X \subseteq R \setminus S \tag{14}$$

we obtain

$$[\![P]\!] \cdot \Phi_p \subseteq \Phi_q \cdot \top \tag{15}$$

equivalent to

$$[\![P]\!] \cdot \Phi_p \subseteq \Phi_q \cdot [\![P]\!]$$

which shares the same scheme as

$$f \cdot \Phi_p \ \subseteq \ \Phi_q \cdot f$$

earlier on.

Summary

In general, we will write "type declaration"

$$\Psi \xleftarrow{R} \Phi \tag{16}$$

to mean

$$R \cdot \Phi \subseteq \Psi \cdot R \tag{17}$$

In words:

- Notation (16) can be regarded as the **type assertion** that, if fed with values (or starting on states) "of type Φ" computation R yields results (moves to states) "of type Ψ" (if it terminates).
- So functional ESC POs and Hoare triples are one and the same device: a way to **type** computations, be them specified as (always terminating, deterministic) functions or encoded into (possibly non-terminating, non-deterministic) programs.

The invariant maintenance (IM) PO

Pointfree:

$$\Phi_{inv} \xleftarrow{R} \Phi_{inv} \tag{18}$$

that is,

$$R \cdot \Phi_{inv} \quad \subseteq \quad \Phi_{inv} \cdot R \tag{19}$$

Pointwise (functions):

$$\langle \forall\ a\ :\ inv\ a :\ inv(f\ a)\rangle \tag{20}$$

Pointwise (relations):

$$\langle \forall\ a\ :\ inv\ a :\ \langle \forall\ a'\ :\ a'\ R\ a :\ inv\ a'\rangle\rangle \tag{21}$$

Mid point: pre-conditioned functions

The most typical situation corresponds to R being a function restricted by some precondition:

- Let $R := f \cdot \Phi_{pre}$ in (18), where pre is a given precondition.
- Then (18) becomes

$$
\begin{array}{ll}
& \Phi_{inv} \xleftarrow{f \cdot \Phi_{pre}} \Phi_{inv} \\
\equiv & \qquad \{ \text{ definition } \} \\
& f \cdot \Phi_{pre} \cdot \Phi_{inv} \subseteq \Phi_{inv} \cdot \top \\
\equiv & \qquad \{ \text{ definition } \} \\
& \Phi_{inv} \xleftarrow{f} \Phi_{pre} \cdot \Phi_{inv} \\
\equiv & \qquad \{ \text{ going pointwise } \} \\
& \langle \forall\ a\ ::\ pre\ a \wedge inv\ a \Rightarrow inv(f\ a) \rangle
\end{array} \tag{22}
$$

Calculating Preconditions for IM

- Very often f and inv are given and pre is the "unknown": the idea is to find pre which is "enough" for (22) to hold.
- In fact, wherever f does not ensure maintenance of invariant inv, there is always a **pre-condition** pre which enforces this at the cost of *partializing* f: in the limit, pre is the *everywhere false* predicate.
- As a rule, the average programmer will become aware of such a pre-condition at runtime, in the **testing** phase.
- One can find it much earlier, at specification time, when trying to discharge the standard proof obligation (22).

PF-ESC instead of invent & verify

However,

- Bound to **invent** pre, we'll hope to have guessed the **weakest** such pre-condition. Otherwise, future use of f will be spuriously constrained.
- Can we be sure of having hit the **weakest** pre-condition?

Our approach (**PF-ESC**) will be as follows:

- We take the PF-transform of $inv(f\ a)$ in (22) — at data level — and attempt to rewrite it to a term involving $inv\ a$ and possibly "something else": the **calculated** pre-condition.
- This will be the weakest provided the calculation stays within equivalence steps (as shown in the next slides).

Weakest pre-conditions

- Let us strengthen (22) to equivalence

$$\langle \forall\, a \;::\; (pre\ a) \wedge (inv\ a) \equiv inv(f\ a) \rangle \tag{23}$$

which PF-transforms to equality

$$\Phi_{pre} \cdot \Phi_{inv} = \top \cdot \Phi_{inv} \cdot f \tag{24}$$

- Later on we will show that (24) ensures *pre* as the **weakest** (up to logical equivalence) pre-condition for *inv* to be preserved.
- Weakest = sufficient + necessary for $inv(f\ a)$ to hold.

Case study 1: PF-ESC at work

We want to calculate the WP for

$$add\ x\ l \;\;\triangleq\;\; x : l$$

to preserve the **no duplicates** invariant on finite lists.

- First step: PF-transform $X^{\star}$ to $\mathbb{N} \rightharpoonup X$ (**simple** relation telling which elements take which position in list).

  Then the **no duplicates** invariant on L is encoded as $\ker L \subseteq id$ (L is injective)

  Finally, $add\ x\ L$ PF-transforms to

$$\underline{x} \cdot \underline{1}^{\circ} \cup L \cdot succ^{\circ} \qquad (25)$$

  cf. back to points: $\{1 \mapsto x\} \cup \{i+1 \mapsto (L\ i) : i \leftarrow \delta\ L\}$.

Case study 1: PF-ESC at work

- Second step: we start from the right hand side $inv(add\ x\ L)$ of (23) and re-write it by successive equivalence steps until we reach:
  - condition $inv\ l$ ...
  - ... "plus something else" — the calculated weakest pre-condition.
- Since the PF-transformed proof has to do with injectivity of union of relations, the following fact

$$R \cup S \text{ is injective } \equiv \qquad\qquad R \text{ is injective} \wedge S \text{ is injective} \wedge R^{\circ} \cdot S \subseteq id \qquad (26)$$

(easy to prove) is likely to be of use.

Case study 1: PF-ESC at work

$add\ x\ L$ has no duplicates

$\equiv \quad \{$ cf. (25) etc $\}$

$\underline{x} \cdot \underline{1}^\circ \cup L \cdot succ^\circ$ is injective

$\equiv \quad \{$ (26) $\}$

$\underline{x} \cdot \underline{1}^\circ$ is injective $\wedge\, L \cdot succ^\circ$ is injective $\wedge\, (\underline{x} \cdot \underline{1}^\circ)^\circ \cdot L \cdot succ^\circ \subseteq id$

$\equiv \quad \{$ definition of injective (twice) ; "al-djabr" (59) $\}$

$\underline{1} \cdot \underline{x}^\circ \cdot \underline{x} \cdot \underline{1}^\circ \subseteq id \wedge succ \cdot L^\circ \cdot L \cdot succ^\circ \subseteq id \wedge \underline{x}^\circ \cdot L \subseteq \underline{1}^\circ \cdot succ$

$\equiv \quad \{$ "al-djabr" (59,60) as much as possible $\}$

$\underline{x}^\circ \cdot \underline{x} \subseteq \underline{1}^\circ \cdot \underline{1} \wedge L^\circ \cdot L \subseteq succ^\circ \cdot succ \wedge \underline{x}^\circ \cdot L \subseteq \underline{1}^\circ \cdot succ$

$\equiv \quad \{$ kernel of constant function is $\top$; *succ* is an injection $\}$

$\textsc{True} \wedge L^\circ \cdot L \subseteq id \wedge \underline{x}^\circ \cdot L \subseteq \underline{1}^\circ \cdot succ$

Case study 1: summary

We have thus calculated:

$$\textit{add}\ x\ L \text{ has no duplicates} \quad \equiv \quad \underbrace{L \text{ is injective}}_{\text{no duplicates in } L} \ \wedge\ \underbrace{\underline{x}^\circ \cdot L \subseteq \underline{1}^\circ \cdot \textit{succ}}_{\text{WP}}$$

PW-expansion of the calculated WP:

$$
\begin{array}{ll}
& \underline{x}^\circ \cdot L \subseteq \underline{1}^\circ \cdot \textit{succ} \\
\equiv & \quad \{ \text{ go pointwise: (48) twice } \} \\
& \langle \forall\ n\ ::\ x\ L\ n \Rightarrow 1 = 1 + n \rangle \\
\equiv & \quad \{ \ L \text{ models list } l \ \} \\
& \langle \forall\ n\ :\ n \in \text{inds}\, l :\ x = (l\ n) \Rightarrow 1 = 1 + n \rangle \\
\equiv & \quad \{ \ 1 = 1 + n \text{ always false } (n \in \mathbb{N}) \ \} \\
& \langle \forall\ n\ :\ n \in \text{inds}\, l :\ (l\ n) \neq x \rangle
\end{array}
$$

Case study 2: PF-ESC at work

From the mobile phone directory problem we select maintenance of the no duplicates invariant by function

$$store\ x \quad \triangleq \quad (take\ 10) \cdot (x\ :) \cdot filter(x \neq)$$

Remarks:

- It's sufficient to show that $(x\ :) \cdot filter(x \neq)$ preserves injectivity, since $take\ n\ L \ \subseteq\ L$ $(\forall n)$ and *smaller than injective is injective*
- Defined over PF-transformed lists, $filter$ becomes

$$filter(x \neq)L \quad \triangleq \quad (\neg\rho\, \underline{x}) \cdot L \tag{27}$$

where the negated range operator $(\neg\rho)$ satisfies property

$$\Phi \subseteq \neg\rho\, R \quad \equiv \quad \Phi \cdot R \subseteq \bot \tag{28}$$

Case study 2: PF-ESC at work

$$
\begin{array}{ll}
& x : (\mathit{filter}(x \neq) L) \text{ is injective} \\
\equiv & \quad \{ \text{ case study 1, (27) } \} \\
& (\neg \rho\, \underline{x}) \cdot L \text{ is injective} \wedge \underline{x}^{\circ} \cdot (\neg \rho\, \underline{x}) \cdot L \subseteq \underline{1}^{\circ} \cdot \mathit{succ} \\
\Leftarrow & \quad \{ \text{ smaller than injective is injective } \} \\
& L \text{ is injective} \wedge \underline{x}^{\circ} \cdot (\neg \rho\, \underline{x}) \cdot L \subseteq \underline{1}^{\circ} \cdot \mathit{succ} \\
\equiv & \quad \{ \text{ converses } \} \\
& L \text{ is injective} \wedge L^{\circ} \cdot (\neg \rho\, \underline{x}) \cdot \underline{x} \subseteq \mathit{succ}^{\circ} \cdot \underline{1} \\
\equiv & \quad \{ (\neg \rho\, \underline{x}) \cdot \underline{x} = \bot \text{ by left-cancellation of (28) } \} \\
& L \text{ is injective} \wedge L^{\circ} \cdot \bot \subseteq \mathit{succ}^{\circ} \cdot \underline{1} \\
\equiv & \quad \{ \text{ bottom is below anything } \} \\
& L \text{ is injective} \wedge \textsc{True}
\end{array}
$$

Case study 2: PF-ESC at work

Moral of this case study:

> *Although the implication in the second step of the reasoning could put weakness of calculated pre-condition at risk, we've calculated the weakest of all conditions anyway (TRUE).*

Exercise 5: Show that (28) stems from "al-djabr" rule

$$\Phi \subseteq \neg\delta\, R \quad \equiv \quad R \subseteq \bot/\Phi \tag{29}$$

among others.

□

Exercise 6: Prove (26).

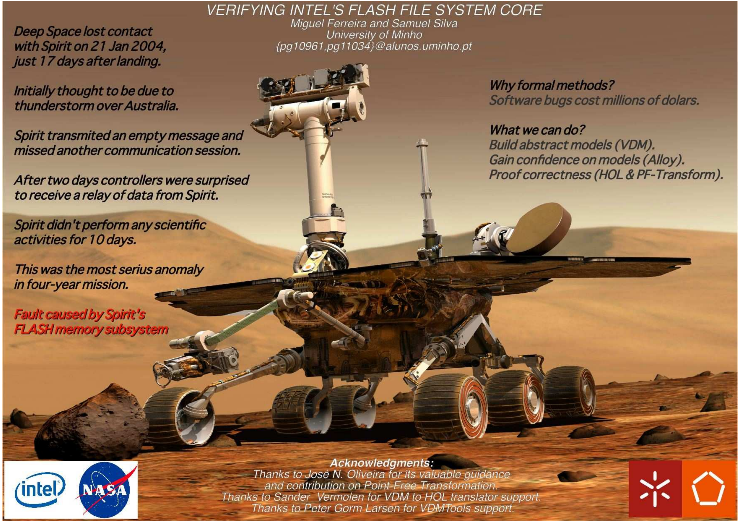
□

Case study 3: Verified File System

A real-life case study:

- **VSR** (Verified Software Repository) initiative
- **VFS** (Verified File System) on Flash Memory — challenge put forward by Rajeev Joshi and Gerard Holzmann (NASA JPL) [2]
- Two levels — POSIX level and (NAND) flash level
- Working document: **Intel ® Flash File System Core Reference Guide** (Oct. 2004) is POSIX aware.

Case study 3: Verified File System



Case study 3: Verified File System

The problem (sample):

File System API Reference

4.6 FS_DeleteFileDir

Deletes a single file/directory from the media

Syntax

```
FFS_Status    FS_DeleteFileDir (
        mOS_char      *full_path,
        UINT8         static_info_type );
```

Parameters

Parameter	Description
*full_path	(IN) This is the full path of the filename for the file or directory to be deleted.
static_info_type	(IN) This tells whether this function is called to delete a file or a directory.

Error Codes/Return Values

FFS_StatusSuccess	Success
FFS_StatusNotInitialized	Failure
FFS_StatusInvalidPath	Failure
FFS_StatusInvalidTarget	Failure
FFS_StatusFileStillOpen	Failure

Verified File System Project

Sample of model's data types (simplified):

$$System = \{table : OpenFileDescriptorTable, tar : Tar\}$$
$$\textbf{inv} \;\; sys \triangleq \langle \forall\, ofd \; : \; ofd \in rng\ (table\ sys) : \; path\ ofd \in dom\ tar\ sys \rangle$$

where

$$OpenFileDescriptorTable = FileHandler \rightharpoonup OpenFileDescriptor$$

$$Tar = Path \rightharpoonup File$$
$$\textbf{inv} \;\; tar \triangleq \langle \forall\, p \; : \; p \in dom\ tar : \; dirName(p) \in dom\ tar \land fileType(attributes(tar(dirName\ p))) = Directory \rangle$$

$$OpenFileDescriptor = \{path : Path, \ldots\}$$

Verified File System Project

(Sample) API function:

$$FS\_DeleteFileDir : Path \rightarrow System \rightarrow (System \times FFS\_Status)$$

$$FS\_DeleteFileDir\ p\ sys \triangleq$$

$$\quad if\ \ p \neq Root \wedge p \in dom\ (tar\ \ sys) \wedge \text{pre-}FS\_DeleteFileDir\_System\ p\ sys$$

$$\quad then\ (FS\_DeleteFileDir\_System\ p\ sys, FFS\_StatusSuccess)$$

$$\quad else\ (sys, FS\_DeleteFileDir\_Exception\ p\ sys)$$

where

$$FS\_DeleteFileDir\_System : Path \rightarrow System \rightarrow System$$

$$FS\_DeleteFileDir\_System\ p\ (h, t) \triangleq$$

$$\quad (h, FS\_DeleteFileDir\_Tar\ \{p\}\ t)$$

$$\mathbf{pre} \left\langle \begin{array}{c} \forall\ buffer \\ buffer \in rng\ \ h : \\ path\ buffer \neq p \wedge \text{pre-}FS\_DeleteFileDir\_Tar\ \ p\ \ t \end{array} \right\rangle$$

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Sample API function (continued):

$$FS\_DeleteFileDir\_Tar : \mathcal{P}\,Path \rightarrow Tar \rightarrow Tar$$
$$FS\_DeleteFileDir\_Tar\ s\ t \triangleq\ tar \setminus s$$
$$\textbf{pre}\ \langle \forall\ p\ :\ p \in dom\ tar :\ dirName\ p \in s \Rightarrow p \in s \rangle;$$

where

$$dirName : Path \rightarrow Path$$
$$dirName\ p \triangleq\ if\ p = Root \vee len\ p = 1$$
$$then\ Root$$
$$else\ blast\ p$$

and so on. (**NB:** *blast* selects all but the last element of a list.)

Invariant structural synthesis (coreflexives)

- Real-size problems show **where complexity is**, namely the intricate structure involving nested datatype invariants.
- Need to calculate the associated coreflexives.
- Denoting by A_p the fact that datatype A is constrained by invariant p, we will write $\in_{A_p}$ to denote the associated coreflexive, calculated by induction on the structure of types:

$$\in_X = id \tag{30}$$

$$\in_{K_p} = \Phi_p \tag{31}$$

$$\in_{(A \times B)_p} = (\in_A \times \in_B) \cdot \Phi_p \tag{32}$$

$$\in_{(A+B)_{[p\ ,q]}} = \in_A \cdot \Phi_p + \in_B \cdot \Phi_q \tag{33}$$

$$\in_{(\mathsf{F}\,A)_p} = \mathsf{F}(\in_A) \cdot \Phi_p \tag{34}$$

Invariant structural synthesis (coreflexives)

Example:

$$
\begin{array}{ll}
& \in_{System} \\
= & \qquad \{ \text{ (32), for } ri \text{ (= "referential integrity") the top level inv. } \} \\
& (\in_{OpenFileDescriptorTable} \times \in_{Tar}) \cdot \Phi_{ri} \\
= & \qquad \{ \text{ } OpenFileDescriptorTable \text{ has no invariant } \} \\
& (id \times \in_{Tar}) \cdot \Phi_{ri} \\
= & \qquad \{ \text{ (31) for } pc \text{ (= "prefix closed") denoting } Tar\text{'s invariant } \} \\
& (id \times \Phi_{pc}) \cdot \Phi_{ri}
\end{array}
\tag{35}
$$

Facing complexity

Need to "find structure" in the specification text:

- *FS_DeleteFileDir p* has conditional "shape"

$$c \longrightarrow \langle f \cdot \Phi_p, \underline{k} \rangle, \langle id, g \rangle \qquad (36)$$

where

- c is the (main) if-then-else's condition
- f abbreviates *FS_DeleteFileDir_System p*
- p is the precondition of f
- k abbreviates *FFS_StatusSuccess*
- g abbreviates *FS_DeleteFileDir_Exception p*

What's the advantage of pattern (36)?

See the "divide and conquer" rules which follow:

Breaking complexity of POs

Further to (5), (7), (9):

- **Trivial**:

$$id \xleftarrow{R} \Phi \quad \equiv \quad \Phi \xleftarrow{R} \bot \quad \equiv \quad \Psi \xleftarrow{\bot} \Phi \quad \equiv \quad \textsc{True} \qquad (37)$$

- **Trading**:

$$\Upsilon \xleftarrow{R} \Phi \cdot \Psi \quad \equiv \quad \Upsilon \xleftarrow{R \cdot \Phi} \Psi \qquad (38)$$

- **Composition** (**Fusion**):

$$\Psi \xleftarrow{R \cdot S} \Phi \quad \Leftarrow \quad \Psi \xleftarrow{R} \Upsilon \wedge \Upsilon \xleftarrow{S} \Phi \qquad (39)$$

Breaking complexity of POs

- **Split by conjunction**:

$$\Psi_1 \cdot \Psi_2 \xleftarrow{R} \Phi \quad \equiv \quad \Psi_1 \xleftarrow{R} \Phi \;\wedge\; \Psi_2 \xleftarrow{R} \Phi \tag{40}$$

— generalizes (7)

- **Weakening/strengthening**:

$$\Psi \xleftarrow{R} \Phi \quad \Leftarrow \quad \Psi \supseteq \Theta \wedge \;\Theta \xleftarrow{R} \Upsilon \;\wedge \Upsilon \supseteq \Phi \tag{41}$$

- **Separation**:

$$\Upsilon \cdot \Theta \xleftarrow{R} \Phi \cdot \Psi \quad \Leftarrow \quad \Upsilon \xleftarrow{R} \Phi \;\wedge\; \Theta \xleftarrow{R} \Psi \tag{42}$$

— outcome of (41), (40)

Breaking complexity of POs

- **Splitting** (functions):

$$\Psi \times \Upsilon \xleftarrow{\langle f,g \rangle} \Phi \quad \equiv \quad \Psi \xleftarrow{f} \Phi \ \wedge \ \Upsilon \xleftarrow{g} \Phi \tag{43}$$

- **Splitting** (in general):

$$\Psi \times \Upsilon \xleftarrow{\langle R,S \rangle} \Phi \quad \equiv \quad \Psi \xleftarrow{R} \Phi \cdot \delta\, S \ \wedge \ \Upsilon \xleftarrow{S} \Phi \cdot \delta\, R \tag{44}$$

- **Product**:

$$\Phi' \times \Psi' \xleftarrow{R \times S} \Phi \times \Psi \quad \equiv \quad \Phi' \xleftarrow{R} \Phi \ \wedge \ \Psi' \xleftarrow{S} \Psi \tag{45}$$

Breaking complexity of POs

- **Conditional**:

$$\psi \xleftarrow{c \to R, S} \Phi \quad \equiv \quad \psi \xleftarrow{R} \Phi \cdot \Phi_c \ \wedge \ \psi \xleftarrow{S} \Phi \cdot \Phi_{\neg c} \qquad (46)$$

  which generalizes (6).

NB:

- Close relationship with Hoare logic axioms

  — but note many **equivalences** instead of implications

Exercise 7: Use the PF-calculus to prove the correctness of the rules given above.

□

Checking *FS_DeleteFileDir*:

$$\in_{System \times FFS\_Status} \xleftarrow{FS\_DeleteFileDir\ \ p} \in_{System}$$

$$\equiv \quad \{\ (36)\ \}$$

$$\in_{System} \times id \xleftarrow{c \rightarrow \langle f \cdot \Phi_p, \underline{k} \rangle, \langle id, g \rangle} \in_{System}$$

$$\equiv \quad \{\ \text{conditional } (46)\ \}$$

$$\in_{System} \times id \xleftarrow{\langle f \cdot \Phi_p, \underline{k} \rangle} \in_{System} \cdot \Phi_c$$

$$\wedge$$

$$\in_{System} \times id \xleftarrow{\langle id, g \rangle} \in_{System} \cdot \Phi_{\neg c}$$

Verified File System Project

$\equiv$ { splitting (44,43) }

$$\in_{System} \xleftarrow{f \cdot \Phi_p} \in_{System} \cdot \Phi_c$$

$$\wedge$$

$$id \xleftarrow{\underline{k}} \in_{System} \cdot \Phi_c \cdot \delta\,(f \cdot \Phi_p)$$

$$\wedge$$

$$\in_{System} \xleftarrow{id} \in_{System} \cdot \Phi_{\neg c}$$

$$\wedge$$

$$id \xleftarrow{g} \in_{System} \cdot \Phi_{\neg c}$$

$\equiv$ { (37), (5) }

Case study 3: Verified File System

$$\in_{System} \xleftarrow{f \cdot \Phi_p} \in_{System} \cdot \Phi_c$$

$\equiv$ { trading (38), unfold $\in_{System}$ (35) }

$$(id \times \Phi_{pc}) \cdot \Phi_{ri} \xleftarrow{f \cdot \Phi_p \cdot \Phi_c} (id \times \Phi_{pc}) \cdot \Phi_{ri}$$

$\Leftarrow$ { separating (42) }

$$\Phi_{ri} \xleftarrow{f \cdot \Phi_p \cdot \Phi_c} \Phi_{ri} \quad \wedge \quad id \times \Phi_{pc} \xleftarrow{f \cdot \Phi_p \cdot \Phi_c} id \times \Phi_{pc}$$

$\equiv$ { trading (38) and implication $c \Rightarrow p$ }

$$\Phi_{ri} \xleftarrow{f} \Phi_{ri} \cdot \Phi_c \quad \wedge$$

$$id \times \Phi_{pc} \xleftarrow{f} (id \times \Phi_{pc}) \cdot \Phi_c$$

Case study 3: Verified File System

- So much for PO calculation "in-the-large".
- Going "in-the-small" means spelling out invariants, functions and pre-conditions and reason as in the previous case studies
- Let us pick the first PO, $\Phi_{ri} \xleftarrow{f} \Phi_{ri} \cdot \Phi_c$, for example.
- As earlier on, we go pointwise and try to rewrite $ri(f(M, N))$ — M keeps open file descriptors, N the file contents — into $ri(M, N)$ + a weakest precondition; then we compare the outcome with what the designer wrote (Φ_c).

Case study 3: Verified File System

Clearly, from slide 25 we infer

$$ri(M, N) \quad \triangleq \quad \rho\,(path \cdot M) \subseteq \delta\, N$$

cf. diagram

$$\begin{tikzcd}
FileHandler \arrow[r, harpoon, "M"] & OpenFileDescriptor \arrow[dl, "path"] \\
Path \arrow[r, harpoon, "N"'] & File
\end{tikzcd}$$

which is a referential integrity constraint relating paths in open-file descriptors and paths in the file store N. PF calculation will lead to

$$ri(M, N) \;\; \triangleq \;\; path \cdot M \subseteq N^\circ \cdot \top \tag{47}$$

thanks to (64) etc.

Case study 3: Verified File System

We calculate:

$$
\begin{array}{ll}
 & ri(f(M,N)) \\
= & \qquad \{\ (36)\ \} \\
 & ri(FS\_DeleteFileDir\_System\ p\ (M,N)) \\
= & \qquad \{\ FS\_DeleteFileDir\_Tar\ s\ t \triangleq\ tar \setminus s \text{ etc }\ \} \\
 & ri(M, N \cdot \neg\rho\,\underline{p})
\end{array}
$$

We can generalize from single path p to a set S of paths:

$$
\begin{array}{ll}
 & ri(M, N \cdot \Phi_{\neg S}) \\
\equiv & \qquad \{\ (47)\ \} \\
 & path \cdot M \subseteq (N \cdot \Phi_{\neg S})^{\circ} \cdot \top \\
\equiv & \qquad \{\ \text{converses } (50,51,\ 54)\ \}
\end{array}
$$

Case study 3: Verified File System

$$
\begin{array}{ll}
 & path \cdot M \subseteq \Phi_{\neg S} \cdot N^{\circ} \cdot \top \\
\equiv & \qquad \{ \text{ (66), coreflexives (55), } (\cdot\top) \text{ distribution } \} \\
 & path \cdot M \subseteq \Phi_{\neg S} \cdot \top \cap N^{\circ} \cdot \top \\
\equiv & \qquad \{ \text{ } \cap\text{-universal (52) } \} \\
 & path \cdot M \subseteq \Phi_{\neg S} \cdot \top \ \wedge \ path \cdot M \subseteq N^{\circ} \cdot \top \\
\equiv & \qquad \{ \text{ “al-djabr” ; (47) } \} \\
 & \underbrace{M \subseteq path^{\circ} \cdot \Phi_{\neg S} \cdot \top}_{\text{wp}} \qquad \wedge \, ri(M, N) \\
\equiv & \qquad \{ \text{ going pointwise } \} \\
 & \langle \forall \ b \ : \ b \in rng \ M : \ path \ b \notin S \rangle \ \wedge \ ri(M, N)
\end{array}
$$

Summary

- Thus we've checked (part) of the pre-condition of *FS_DeleteFileDir_System*, recall slide 25
- The other checks are performed in a similar way.
- Two levels of PO calculation: **in-the-large** (PO level) and **in-the-small** (where PF-notation describes data).
- PO-level useful in preparing POs for a theorem prover, recall diagram of slide 3.

Background

"Napkin" rule:

$$b(f^\circ \cdot R \cdot g)a \quad \equiv \quad (f\ b)R(g\ a) \tag{48}$$

Converses:

$$\begin{aligned}
(R \cup S)^\circ &= R^\circ \cup S^\circ && (49)\\
(R \cdot S)^\circ &= S^\circ \cdot R^\circ && (50)\\
(R^\circ)^\circ &= R && (51)
\end{aligned}$$

Background

Meet and join:

$$X \subseteq R \cap S \equiv X \subseteq R \wedge X \subseteq S \tag{52}$$

$$R \cup S \subseteq X \equiv R \subseteq X \wedge S \subseteq X \tag{53}$$

Coreflexives are **symmetric** and **transitive**:

$$\Phi^\circ = \Phi = \Phi \cdot \Phi \tag{54}$$

Meet of two coreflexives is composition:

$$\Phi \cap \Psi = \Phi \cdot \Psi \tag{55}$$

Background

Equality on relations $B \xleftarrow{R,S} A$:

$$R = S \quad \equiv \quad R \subseteq S \wedge S \subseteq R \tag{56}$$

Alternative to (56) — **indirect equality** rules:

$$R = S \quad \equiv \quad \langle \forall\, X \;::\; (X \subseteq R \equiv X \subseteq S)\rangle \tag{57}$$

$$\equiv \quad \langle \forall\, X \;::\; (R \subseteq X \equiv S \subseteq X)\rangle \tag{58}$$

Shunting rules:

$$f \cdot R \subseteq S \quad \equiv \quad R \subseteq f^{\circ} \cdot S \tag{59}$$

$$R \cdot f^{\circ} \subseteq S \quad \equiv \quad R \subseteq S \cdot f \tag{60}$$

Therefore

$$f \cdot (R \cup S) \quad = \quad f \cdot R \cup f \cdot S \tag{61}$$

Background

Kernel, image (the same for respectively δ, ρ):

$$\ker(R^\circ) = \operatorname{img} R \tag{62}$$

$$\operatorname{img}(R^\circ) = \ker R \tag{63}$$

Range:

$$\rho R \subseteq \Phi \equiv R \subseteq \Phi \cdot \top \tag{64}$$

Domain/range elimination:

$$\top \cdot \delta R = \top \cdot R \tag{65}$$

$$\rho R \cdot \top = R \cdot \top \tag{66}$$

Background

Weakest (liberal) pre-condition is the upper adjoint of the following "al-djabr" rule [1] which combines two already seen — range (64) and left division (13):

$$\rho\,(R \cdot \Phi) \subseteq \Psi \quad \equiv \quad \Phi \subseteq R \setminus \Psi \tag{67}$$

The pointwise version $wlp\ R\ \psi$ of $R \setminus \Psi$ is:

$$wlp\ R\ \psi \quad \triangleq \quad \langle \bigvee\ \phi\ ::\ \langle \forall\ b, a\ :\ b\ R\ a :\ \phi\ a \Rightarrow \psi\ b \rangle\rangle$$

In the slide which follows we show that, if equivalence (23) holds then *pre* is the weakest precondition for *inv* to be maintained. The calculation proceeds by **indirect equality** (57) over coreflexive Φ:

Weakest pre-conditions

$$\Phi \subseteq \Phi_{pre} \cdot \Phi_{inv}$$

$\equiv$ { rep. equal by equals (24) }

$$\Phi \subseteq \top \cdot \Phi_{inv} \cdot f$$

$\equiv$ { "al-djabr" rule (59) ; converses }

$$f \cdot \Phi \subseteq \Phi_{inv} \cdot \top$$

$\equiv$ { range (64) }

$$\rho\,(f \cdot \Phi) \subseteq \Phi_{inv}$$

$\equiv$ { weakest pre-condition (67) }

$$\Phi \subseteq f \mathbin{\backslash\!\!\!\bullet} \Phi_{inv}$$

$::$ { indirection (57) }

$$\Phi_{pre} \cdot \Phi_{inv} = f \mathbin{\backslash\!\!\!\bullet} \Phi_{inv}$$

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