Coalgebra and coinduction (II)

L.S. Barbosa

Dept. Informática,
Universidade do Minho
Braga, Portugal

DI-CCTC, UM, 2008

- Moore and Mealy transducers
- Behavioural effects
- The general case: coalgebras
- Bisimulation and bisimilarity

Moore transducers

state space	U
transition function	$\overline{\mathsf{nx}} : U^A \longleftarrow U$
attribute (or label)	$\mathsf{at} : B \longleftarrow U$

i.e.,

$$p \;=\; \langle \overline{\mathsf{nx}}, \mathsf{at} \rangle : U^A \times B \longleftarrow U$$

Notation:

$$u \xrightarrow{a}_p u' \quad \Leftrightarrow \quad \overline{\mathsf{nx}}\, u\, a \;=\; u'$$

$$u \downarrow_p b \quad \Leftrightarrow \quad \mathsf{at}\, u \;=\; b$$

Moore transducers

The behaviour of p at (from) a state $u \in U$ is revealed by successive observations (experiments) triggered on input of different values $a \in A$:

$$[\![(p)]\!]\ u \quad = \quad [\mathsf{at}\ u, \mathsf{at}\ (\overline{\mathsf{nx}}\ u\ a_0), \mathsf{at}\ (\overline{\mathsf{nx}}\ (\overline{\mathsf{nx}}\ u\ a_0)\ a_1), \ldots]$$

$$\begin{array}{rcl} [\![(p)]\!] u\ \underline{\mathsf{nil}} & = & \mathsf{at}\, u \\ [\![(p)]\!] u\,(a : t) & = & [\![(p)]\!]\,(\overline{\mathsf{nx}}\ u\ a)\ t \end{array}$$

which means that

Moore behaviours are elements of B^{A^*}
(depicted as rooted trees whose branches are labelled by sequences of inputs and leaves by B values)

Moore morphisms

A morphism

$$h : q \longleftarrow p$$

where

$$\begin{aligned} p &= \langle \overline{\mathsf{nx}}, \mathsf{at} \rangle : U^A \times B \longleftarrow U \\ q &= \langle \overline{\mathsf{nx}}', \mathsf{at}' \rangle : V^A \times B \longleftarrow V \end{aligned}$$

is a function $h : V \longleftarrow U$ such that

$$\begin{tikzcd} U \arrow[r, "p"] \arrow[d, "h"'] & U^A \times B \arrow[d, "h^A \times \mathrm{id}"] \\ V \arrow[r, "q"] & V^A \times B \end{tikzcd}$$

To avoid the explicit use of exponentials, the diagram can be decomposed into:

Moore morphisms

$$\begin{tikzcd} U \arrow[r, "\mathsf{at}"] \arrow[d, "h"'] & B \arrow[d, "\mathsf{id}"] \\ V \arrow[r, "\mathsf{at}'"] & B \end{tikzcd}$$

and

$$\begin{tikzcd} U \times A \arrow[r, "nnx"] \arrow[d, "h\times\mathsf{id}"'] & U \arrow[d, "h"] \\ V \times A \arrow[r, "\mathsf{nx}'"] & V \end{tikzcd}$$

corresponding to

$$\begin{aligned} \mathsf{at}' \cdot h \quad &= \ \mathsf{at} \\ \mathsf{nx}' \cdot (h \times \mathsf{id}) \quad &= \ h \cdot \mathsf{nx} \end{aligned}$$

Moore morphisms

Clearly, morphisms preserve attributes and transitions

$$
\begin{array}{ll}
& u \xrightarrow{a}_p u' \quad \text{and} \quad u \downarrow_p b \\
\Leftrightarrow & \qquad \{ \text{ definition } \} \\
& \mathsf{nx}(u, a) = u' \quad \text{and} \quad \mathsf{at}\, u = b \\
\Leftrightarrow & \qquad \{ \text{ Liebniz } \} \\
& h\, \mathsf{nx}(u, a) = h\, u' \quad \text{and} \quad \mathsf{at}\, u = b \\
\Leftrightarrow & \qquad \{ \, h \text{ is a morphism } \} \\
& \mathsf{nx}'(h\, u, a) = h\, u' \quad \text{and} \quad \mathsf{at}'\, h\, u = b \\
\Leftrightarrow & \qquad \{ \text{ definition } \} \\
& h\, u \xrightarrow{a}_q h\, u' \quad \text{and} \quad h\, u \downarrow_q b
\end{array}
$$

The final Moore transducer

Moore behaviours organise themselves into a final Moore machine over B^{A^*}

$$\omega = \langle \overline{\mathsf{nx}}_\omega, \mathsf{at}_\omega \rangle : (B^{A^*})^A \times B \longleftarrow B^{A^*}$$

where

$$\begin{array}{lcll} \mathsf{at}_\omega\, f & = & f\ \mathsf{nil} & \text{ie, the value before any input} \\ \overline{\mathsf{nx}}_\omega\, f\ a & = & \lambda\, s\, .\ f\,(a : s) & \text{every input determines its evolution} \end{array}$$

The final Moore transducer

Th: Coalgebra ω is the final coalgebra for $\mathsf{T}\,X = X^A \times B$

because

1. For any $p = \langle \overline{\mathsf{nx}}, \mathsf{at} \rangle$, $[\![(p)]\!]$ is a Moore morphism $[\![(p)]\!] : \omega \longleftarrow p$

$$
\begin{array}{cl}
 & \mathsf{at}_\omega \cdot [\![(p)]\!] \;=\; \mathsf{at} \\
\Leftrightarrow & \quad \{ \text{ introduction of variables } \} \\
 & \mathsf{at}_\omega([\![(p)]\!]\, u) \;=\; \mathsf{at}\, u \\
\Leftrightarrow & \quad \{ \text{ definition of } \mathsf{at}_\omega \; \} \\
 & ([\![(p)]\!]\, u)\, \mathsf{nil} \;=\; \mathsf{at}\, u \\
\Leftrightarrow & \quad \{ \text{ definition of } [\![(p)]\!] \; \} \\
 & \textsc{True}
\end{array}
$$

The final Moore transducer

$$
\begin{aligned}
& \mathsf{nx}_\omega \cdot (\llbracket (p) \rrbracket \times \mathsf{id}) \; = \; \llbracket (p) \rrbracket \cdot \mathsf{nx} \\
\Leftrightarrow \quad & \qquad \{ \text{ introduction of variables and application } \} \\
& \mathsf{nx}_\omega (\llbracket (p) \rrbracket \, u, a) \; = \; \llbracket (p) \rrbracket \, \mathsf{nx} \, (u, a) \\
\Leftrightarrow \quad & \qquad \{ \text{ definition of } \mathsf{nx}_\omega \; \} \\
& \lambda \, s \, . \;\; (\llbracket (p) \rrbracket \, u) \, (a : s) \; = \; \llbracket (p) \rrbracket \, \mathsf{nx} \, (u, a) \\
\Leftrightarrow \quad & \qquad \{ \text{ introduction of variables and application } \} \\
& (\llbracket (p) \rrbracket \, u) \, (a : t) \; = \; (\llbracket (p) \rrbracket \, \mathsf{nx} \, (u, a)) \, t \\
\Leftrightarrow \quad & \qquad \{ \text{ definition of } \llbracket (p) \rrbracket \; \} \\
& \textsc{True}
\end{aligned}
$$

The final Moore transducer

2\. ... and is unique

Exercise. Prove uniqueness (by induction on A^*)

Instances of Moore transducers

$$Queue = \langle \overline{\mathsf{nx}}, \mathsf{at} \rangle : (E^*)^{E+\mathbf{1}} \times ((E+\mathbf{1}) \times \mathbf{2}) \longleftarrow E^*$$

with

$$\begin{array}{rrcl} & \mathsf{at} & = & \langle \mathsf{top}, \mathsf{isempty?} \rangle \\ \text{where} & \mathsf{top}\ s & = & (s = \underline{\mathsf{nil}} \rightarrow \iota_2 *,\ \iota_1(\mathsf{last}\, s)\ ) \\ & \mathsf{isempty?}\ s & = & s = \underline{\mathsf{nil}} \end{array}$$

$$\begin{array}{rrcl} & \mathsf{nx} & = & [\mathsf{enq}, \mathsf{deq}] \cdot \mathsf{dl} \\ \text{where} & \mathsf{enq}\ (s, e) & = & e : s \\ & \mathsf{deq}\ (s, *) & = & (s = \underline{\mathsf{nil}} \rightarrow s,\ (\mathsf{blast}\, s)) \end{array}$$

Instances of Moore transducers

Make $B = \mathbf{2}$ in $\mathsf{T}\,X = X^A \times B$.
The carrier (or state space) of the corresponding final coalgebra is

$$\mathbf{2}^{A^*} \;\cong\; \mathcal{P}A^*$$

and its dynamics is $\langle \overline{\mathsf{nx}}_\omega, \mathsf{at}_\omega \rangle : (\mathcal{P}A^*)^A \times \mathbf{2} \longleftarrow \mathcal{P}A^*$
where

$$\begin{array}{rcl} \mathsf{at}_\omega\, L & = & \underline{\mathsf{nil}} \in L \\ \overline{\mathsf{nx}}_\omega\, L & = & \lambda\, a\,.\; \{a \in A^* |\; (a : s) \in L\} \end{array}$$

Exercise. ... what are we talking about?
Exercise. Make $A = \mathbf{1}$ in $\mathsf{T}\,X = X^A \times B$. What comes up?

Mealy transducers

state space	U
reactive transition function	$\overline{\mathsf{ac}} : (U \times B)^A \longleftarrow U$

Notation:

$$u \xrightarrow{a/b}_p u' \quad \Leftrightarrow \quad \overline{\mathsf{ac}}\, u\, a = (u', b)$$

Mealy transducers

The behaviour of p at a state $u \in U$ is revealed by successive observations (experiments) triggered on input of different values $a \in A$:

$$[\![p]\!] \; u \quad = \quad [\pi_2(\overline{\mathsf{ac}}\, u\, a_0), \pi_2(\overline{\mathsf{ac}}\,(\pi_2(\overline{\mathsf{ac}}\, u\, a_0))\, a_1, \ldots]$$

$$\begin{array}{rcl} [\![p]\!] u\, [a] & = & \pi_2(\mathsf{ac}\, u\, a) \\ [\![p]\!] u\, (a : t) & = & [\![p]\!]\, (\pi_1(\mathsf{ac}\, u\, a))\, t \end{array}$$

which means that

$$\boxed{\text{Mealy behaviours are elements of } B^{A^+}}$$

Mealy transducers

Mealy behaviours can alternatively be regarded as

causal functions from A^ω to B^ω

A causal function f over streams is such that, for all $s, t \in A^\omega$ and $n \in \mathbb{N}$,

$$\langle \forall\, k \ : \ k \leq n : \ s\, k \ = \ t\, k \rangle \ \Rightarrow \ (f\, s\, n \ = \ f\, t\, n)$$

i.e, the n-th element of $f\, s$ depends only on the first n elements of input stream s

... upon which the final Mealy automata can be defined:

The final Mealy transducer

Mealy behaviours organise themselves into a final Mealy automata over $\Gamma = \{f : B^\omega \longleftarrow A^\omega | \ f \text{ is causal}\}$

$$\overline{\omega} : (\Gamma \times B)^A \longleftarrow \Gamma$$

where

$$\overline{\omega}\, f\, a \quad = \quad \langle \lambda\, s\, .\, \mathsf{tl}\ f(a : s), \mathsf{hd}\, f(a : r) \rangle$$

which means that

- the next state acts as f after a has been seen
- the output $\mathsf{hd}\, f(a : r)$ depends only on f and a; therefore, the tail r of the input stream is irrelevant.

Exercises

Exercise. Characterize Mealy morphisms. Draw the corresponding diagram and derive an equational definition.

Exercise. Prove that the Mealy transducer over Γ defined above is final.

- Moore and Mealy transducers
- Behavioural effects
- The general case: coalgebras
- Bisimulation and bisimilarity

Non-determinism

Further behavioural effects can be introduced in the basic machines discussed so far by 'sophisticating' the corresponding signature functor. For example,

- non-determinism is captured by the powerset functor $\mathcal{P}$

Automata	$\mathsf{T}X = B \times X$	$\mathsf{T}X = \mathcal{P}(B \times X)$
Moore transducer	$\mathsf{T}X = X^A \times B$	$\mathsf{T}X = \mathcal{P}(X)^A \times B$
Mealy transducer	$\mathsf{T}X = (X \times B)^A$	$\mathsf{T}X = \mathcal{P}(X \times B)^A$

Example: non-deterministic automata

Coalgebras

$$p : \mathcal{P}(B \times U) \longleftarrow U$$

as relations

$$P : B \times U \longleftarrow U$$

through the relational transpose

$$p = \Lambda P \Leftrightarrow P = \in \cdot p$$

Notation:

$$(b, x')\, P\, x \Leftrightarrow (b, x') \in p\, x \Leftrightarrow x'\, P_b\, x \Leftrightarrow x \xrightarrow{b}_p x'$$

Example: non-deterministic automata

Th: A morphism between two non-deterministic automata p and q satisfies

$$(\mathrm{id} \times h) \cdot P \;=\; Q \cdot h \tag{1}$$

because

$$
\begin{array}{cl}
 & (\mathrm{id} \times h) \cdot P \;=\; Q \cdot h \\
\Leftrightarrow & \quad \{ \text{ relational transpose is an isomorphism } \} \\
 & \Lambda\,((\mathrm{id} \times h) \cdot P) \;=\; \Lambda\,(Q \cdot h) \\
\Leftrightarrow & \quad \{ \; \Lambda\,(f \cdot R) = \mathcal{P} f \cdot \Lambda\, R \;\text{ and }\; \Lambda\,(R \cdot f) = \Lambda\, R \cdot f \text{ and definition } \} \\
 & \mathcal{P}(\mathrm{id} \times h) \cdot \Lambda\,(\in \cdot\, p) \;=\; \Lambda\,(\in \cdot\, q) \cdot h \\
\Leftrightarrow & \quad \{ \; \Lambda\,(R \cdot f) = \Lambda\, R \cdot f \; \} \\
 & \mathcal{P}(\mathrm{id} \times h) \cdot \Lambda \in \cdot\, p \;=\; \Lambda \in \cdot\, q \cdot h \\
\Leftrightarrow & \quad \{ \; \Lambda \in = \mathrm{id} \; \} \\
 & \mathcal{P}(\mathrm{id} \times h) \cdot p \;=\; q \cdot h
\end{array}
$$

Example: non-deterministic automata

Function p, relation p and the B-indexed family of relations $\{P_b|\ b \in B\}$, all represent the same structure. Therefore, a morphism between non-deterministic automata can be defined by the commutativity of any of the following diagrams (of functions or relations, respectively):

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "p"'] & V \arrow[d, "q"] \\ \mathcal{P}(B \times U) \arrow[r, "\mathcal{P}\mathrm{id} \times h"] & \mathcal{P}(B \times V) \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "h"] \arrow[d, "P"'] & V \arrow[d, "Q"] \\ B \times U \arrow[r, "\mathrm{id} \times h"] & B \times V \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "h"] \arrow[d, "P_b"'] & V \arrow[d, "Q_b"] \\ U \arrow[r, "h"] & V \end{tikzcd}$$

Example: non-deterministic automata

Therefore, equation (1) equivales to

$$\begin{aligned} h \cdot P_b &\subseteq Q_b \cdot h \\ Q_b \cdot h &\subseteq h \cdot P_b \end{aligned}$$

entailing, respectively, preservation of p-transitions and reflection of q-transitions, i.e.,

$$\langle \forall\ u, u' \ :\ u, u' \in U :\ u \xrightarrow{b}_p u' \ \Rightarrow\ h\ u \xrightarrow{b}_q h\ u' \rangle \tag{2}$$

$$\langle \forall\ u, v \ :\ u \in U, v' \in V :\ h\ u \xrightarrow{b}_q v' \ \Rightarrow\ \langle \exists\ u' \ :\ u' \in U :\ u \xrightarrow{b}_p u' \ \wedge\ v' = h\ u' \rangle \rangle \tag{3}$$

because

Example: non-deterministic automata

Proof of (2):

$$h \cdot P_b \subseteq Q_b \cdot h$$

$\Leftrightarrow$ { shunting }

$$P_b \subseteq h^\circ \cdot Q_b \cdot h$$

$\Leftrightarrow$ { PF transform }

$$\langle \forall\ u, u' \ :\ u, u' \in U :\ u' \, P_b \, u \Rightarrow u'(h^\circ \cdot Q_b \cdot h)u \rangle$$

$\Leftrightarrow$ { "guardanapo" rule }

$$\langle \forall\ u, u' \ :\ u, u' \in U :\ u' \, P_b \, u \Rightarrow (h\, u')Q_b(h\, u) \rangle$$

$\Leftrightarrow$ { $P_b = (\xrightarrow{b}_p)^\circ$ }

$$\langle \forall\ u, u' \ :\ u, u' \in U :\ u \xrightarrow{b}_p u' \Rightarrow h\, u \xrightarrow{b}_q h\, u' \rangle$$

Example: non-deterministic automata

Proof of (2):

$$Q_b \cdot h \subseteq h \cdot P_b$$

$\Leftrightarrow$ $\{ \cdot R \vdash /R \}$

$$Q_b \subseteq (h \cdot P_b)/h$$

$\Leftrightarrow$ { definition of left division and PF transform }

$$\langle \forall\ v, v' \ :\ v, v' \in V :\ v' Q_b v \Rightarrow \langle \forall\ u \ :\ u \in U :\ v = h\, u \ \Rightarrow\ v'(h \cdot P_b)u \rangle\rangle$$

$\Leftrightarrow$ { quantifier trading (twice) }

$$\langle \forall\ v, v' \ :\ v, v' \in V \wedge v' Q_b v :\ \langle \forall\ u \ :\ u \in U \wedge v = h\, u :\ v'(h \cdot P_b)u \rangle\rangle$$

$\Leftrightarrow$ { quantifier nesting (twice, in opposite directions) }

$$\langle \forall\ u, v' \ :\ u \in U \wedge v' \in V :\ \langle \forall\ v \ :\ v \in V \wedge v = h\, u \wedge v' Q_b v :\ v'(h \cdot P_b)u \rangle\rangle$$

Example: non-deterministic automata

$$\langle \forall\ u, v' \ :\ u \in U \wedge v' \in V :\ \langle \forall\ v \ :\ v \in V \wedge v = h\,u \wedge v' Q_b v :\ v'(h \cdot P_b) u \rangle\rangle$$

$\Leftrightarrow$ { quantifier trading }

$$\langle \forall\ u, v' \ :\ u \in U \wedge v' \in V :\ \langle \forall\ v \ :\ v = h\,u :\ (v \in V \wedge v' Q_b v) \Rightarrow v'(h \cdot P_b) u \rangle\rangle$$

$\Leftrightarrow$ { quantifier one-point rule }

$$\langle \forall\ u, v' \ :\ u \in U \wedge v' \in V :\ (h\,u \in V \wedge v' Q_b (h\,u)) \Rightarrow v'(h \cdot P_b) u \rangle$$

$\Leftrightarrow$ { h type and definition of relational composition }

$$\langle \forall\ u, v' \ :\ u \in U \wedge v' \in V :\ v' Q_b v \Rightarrow \langle \exists\ u' \ :\ u' \in U :\ v' = h\,u \wedge u' P_b u \rangle\rangle$$

$\Leftrightarrow$ { $P_b = (\xrightarrow{b}_p)^\circ$ }

$$\langle \forall\ u, v' \ :\ u \in U \wedge v' \in V :\ v' Q_b v \Rightarrow \langle \exists\ u' \ :\ u' \in U :\ v' = h\,u \wedge u \xrightarrow{b}_p u' \rangle\rangle$$

Partiality

Automata	$\mathsf{T}X = B \times X$	$\mathsf{T}X = (B \times X) + \mathbf{1}$
Moore transducer	$\mathsf{T}X = X^A \times B$	$\mathsf{T}X = (X + \mathbf{1})^A \times B$
Mealy transducer	$\mathsf{T}X = (X \times B)^A$	$\mathsf{T}X = ((X \times B) + \mathbf{1})^A$

In general: monads introduce behaviour

Automata	$\mathsf{T}X = B \times X$	$\mathsf{T}X = \mathsf{B}(B \times X)$
Moore transducer	$\mathsf{T}X = X^A \times B$	$\mathsf{T}X = \mathsf{B}(X)^A \times B$
Mealy transducer	$\mathsf{T}X = (X \times B)^A$	$\mathsf{T}X = \mathsf{B}(X \times B)^A$

where B is a strong monad capturing a particular behavioural effect.

Behaviour monads

- Partiality: $\mathsf{B} = \mathsf{Id} + \mathbf{1}$
- Non determinism: $\mathsf{B} = \mathcal{P}$
- Ordered non determinism: $\mathsf{B} = \mathsf{Id}^*$
- Monoidal labelling: $\mathsf{B} = \mathsf{Id} \times M$, with M a monoid.
- 'Metric' non determinism: $\mathsf{B} = \mathsf{Bag}_M$ based on $\langle M, \oplus, \otimes \rangle$, where $\otimes$ distributes over $\oplus$, both defining Abelian monoids over M.

Behaviour monads

$$\langle \mathsf{B}, \eta, \mu \rangle$$

where

$$\begin{aligned} \eta &: \mathsf{Id} \Longleftarrow \mathsf{B} && \text{(to make a behavioural annotation)} \\ \mu &: \mathsf{BB} \Longleftarrow \mathsf{B} && \text{(to flatten nested annotations)} \end{aligned}$$

being strong entails the presence of right and left strength for context handling:

$$\begin{aligned} \mathsf{B}(\mathsf{Id} \times -) &: \mathsf{B} \times - \Longleftarrow \mathsf{B} \times - \\ \mathsf{B}(- \times \mathsf{Id}) &: - \times \mathsf{B} \Longleftarrow - \times \mathsf{B} \end{aligned}$$

Behaviour monads

Furthermore, Kleisli compositions

$$\delta r_{I,J} \;=\; \tau_{r_{I,J}} \bullet \tau_{l_{\mathrm{B}I,J}} \quad \text{and} \quad \delta l_{I,J} = \tau_{l_{I,J}} \bullet \tau_{r_{I,\mathrm{B}J}}$$

map

$$\mathrm{B}I \times \mathrm{B}J \;\text{ to }\; \mathrm{B}(I \times J)$$

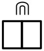
specifying a sort of sequential composition of B-computations

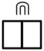
B is a commutative monad if $\delta r_{I,J} \;=\; \delta l_{I,J}$

$\boxed{\text{... plus a handful of equational laws}}$

- Moore and Mealy transducers
- Behavioural effects
- The general case: coalgebras
- Bisimulation and bisimilarity

Algebras

a tool box: 

an assembly process: artifact $\xleftarrow{a}$  artifact

- algebras describe assembly processes
- and abstract data types as (initial) algebras (term algebras)
- emphasis is on construction

Coalgebras

a lens:	$\bigcirc\frown\bigcirc$
an observation structure:	$\bigcirc\frown\bigcirc$ universe $\xleftarrow{c}$ universe

- coalgebras describe observation structures (*i.e.*, transition systems)
- and abstract behaviour types as (final) coalgebras
- emphasis is on observation

Typical lens

- 'opaque'

$$\bigcirc\!\frown\!\bigcirc\; U \;=\; \mathbf{1}$$

- black & white

$$\bigcirc\!\frown\!\bigcirc\; U \;=\; \mathbf{2}$$

- colouring

$$\bigcirc\!\frown\!\bigcirc\; U \;=\; O$$

... in each case the colour set acts as a space classifier

Typical lens

- partiality

$$\bigcirc\!\frown\!\bigcirc\; U \;=\; U + \mathbf{1}$$

- visible attributes

$$\bigcirc\!\frown\!\bigcirc\; U \;=\; O \times U$$

- external stimulus

$$\bigcirc\!\frown\!\bigcirc\; U \;=\; U^{I}$$

- non determinism

$$\bigcirc\!\frown\!\bigcirc\; U \;=\; \mathcal{P}U$$

Question

Which lens shall we seek?

- The main criteria is to choose functors for which the final coalgebra does exist
- Such is the case of the all polynomial functors as well as finite powerset functor

Coalgebras

A coalgebra for a functor T is any function from a set U (its carrier) to $\mathsf{T}U$:

$$\alpha : \mathsf{T}U \longleftarrow U$$

For any functor T, if its space of behaviours can be made a T-coalgebra itself

$$\omega_{\mathsf{T}} : \mathsf{T}\nu_{\mathsf{T}} \longleftarrow \nu_{\mathsf{T}}$$

this is the final coalgebra: from any other T-coalgebra p there is a unique morphism $[\![(p)]\!]$ making the following diagram to commute:

$$\begin{tikzcd}
\nu_{\mathsf{T}} \arrow[r, "\omega_{\mathsf{T}}"] & \mathsf{T}\nu_{\mathsf{T}} \\
U \arrow[u, "{[\![(p)]\!]}"] \arrow[r, "p"'] & \mathsf{T}U \arrow[u, "{\mathsf{T}[\![(p)]\!]}"']
\end{tikzcd}$$

Coalgebras

The universal property is equivalently captured by the following law:

$$k = [\![p]\!] \;\Leftrightarrow\; \omega_{\mathsf{T}} \cdot k = \mathsf{T}\ k \cdot p$$

- Existence $\Leftrightarrow$ definition principle (co-recursion)
- Uniqueness $\Leftrightarrow$ proof principle (co-induction)

From which:

$$\begin{array}{rrcl} \text{cancellation} & \omega_{\mathsf{T}} \cdot [\![p]\!] & = & \mathsf{T}\ [\![p]\!] \cdot p \\ \text{reflection} & [\![\omega_{\mathsf{T}}]\!] & = & \mathrm{id}_{\nu_{\mathsf{T}}} \\ \text{fusion} & [\![p]\!] \cdot h & = & [\![q]\!] \quad \text{if} \quad p \cdot h = \mathsf{T}\ h \cdot q \end{array}$$

Coalgebras

Example: fusion law

$$
\begin{tikzcd}
\nu_{\mathsf{T}} \arrow[r, "\omega_{\mathsf{T}}"] & \mathsf{T}\nu_{\mathsf{T}} \\
U \arrow[u, "\llparenthesis p \rrparenthesis"] \arrow[r, "p"] & \mathsf{T}U \arrow[u, "\mathsf{T}\llparenthesis p \rrparenthesis"'] \\
V \arrow[u, "h"] \arrow[r, "q"] \arrow[uu, bend left=60, "\llparenthesis q \rrparenthesis"] & \mathsf{T}V \arrow[u, "\mathsf{T}h"']
\end{tikzcd}
$$

Coalgebras

Example: fusion law

$$
\begin{array}{ll}
 & [\![(p)]\!] \cdot h = [\![(q)]\!] \\
\Leftrightarrow & \qquad \{ \text{ universal law } \} \\
 & \omega \cdot [\![(p)]\!] \cdot h = \mathsf{T}([\![(p)]\!] \cdot h) \cdot q \\
\Leftrightarrow & \qquad \{ \text{ cancellation law and } T \text{ functor } \} \\
 & \mathsf{T}[\![(p)]\!] \cdot p \cdot h = \mathsf{T}[\![(p)]\!] \cdot \mathsf{T}h \cdot q \\
\Leftarrow & \qquad \{ \text{ function equality } \} \\
 & p \cdot h = \mathsf{T}h \cdot q
\end{array}
$$

Coalgebras

From which one may generalise the fundamental result (proved above for the case of streams)

$\boxed{\text{Th: morphisms preserve behaviour: } [\![(p)]\!] = [\![(q)]\!] \cdot h}$

Proof by coinduction

$\boxed{\text{Example: } \mathsf{map}_f \text{ and generic laws}}$

$$\mathsf{map}_{f \cdot g} = \mathsf{map}_f \cdot \mathsf{map}_g$$

defining map_f as follows:

$$\begin{tikzcd}
B^\omega \arrow[rr, "\omega_B"] & & B \times B^\omega \\
A^\omega \arrow[u, "\mathsf{map}_f"] \arrow[r, "\omega_A"] & A \times A^\omega \arrow[r, "f \times \mathsf{id}"] & A \times B^\omega \arrow[u, "\mathsf{id} \times \mathsf{map}_f"']
\end{tikzcd}$$

Proof by coinduction

$$\mathsf{map}_{f \cdot g} = \mathsf{map}_f \cdot \mathsf{map}_g$$

$$\Leftrightarrow \qquad \{ \text{ map definition } \}$$

$$[\![((f \cdot g) \times \mathsf{id}) \cdot \omega]\!] = [\![(f \times \mathsf{id}) \cdot \omega]\!] \cdot \mathsf{map}_g$$

$$\Leftarrow \qquad \{ \text{ coinduction fusion law } \}$$

$$(f \times \mathsf{id}) \cdot \omega \cdot \mathsf{map}_g = (\mathsf{id} \times \mathsf{map}_g) \cdot ((f \cdot g) \times \mathsf{id}) \cdot \omega$$

$$\Leftrightarrow \qquad \{ \text{ coinduction cancellation law } \}$$

$$(f \times \mathsf{id}) \cdot (\mathsf{id} \times \mathsf{map}_g) \cdot (g \times \mathsf{id}) \cdot \omega = (\mathsf{id} \times \mathsf{map}_g) \cdot ((f \cdot g) \times \mathsf{id}) \cdot \omega$$

$$\Leftrightarrow \qquad \{ \text{ functoriality } \}$$

$$((f \cdot g) \times \mathsf{map}_g) \cdot \omega = ((f \cdot g) \times \mathsf{map}_g) \cdot \omega$$

Proof by coinduction

$\boxed{\text{but this is just an instance of a more general result:}}$

$$\mathsf{map}_{\mathsf{T}}\ (g \cdot f) \ = \ \mathsf{map}_{\mathsf{T}}\ g \cdot \mathsf{map}_{\mathsf{T}}\ f$$

$$\begin{tikzcd}
\nu_{\mathsf{T}_B} \arrow[rr, "\omega_{\mathsf{T}_B}"] & & \mathsf{T}_B\ \nu_{\mathsf{T}_B} \\
\nu_{\mathsf{T}_A} \arrow[u, "\mathsf{map}_{\mathsf{T}}\ f"] \arrow[r, "\omega_{\mathsf{T}_A}"] & \mathsf{T}_A\ \nu_{\mathsf{T}_A} \arrow[r, "\mathsf{T}\ (f, \mathsf{id}_{\mathsf{map}_{\mathsf{T}}\ A})"] & \mathsf{T}_B\ \nu_{\mathsf{T}_A} \arrow[u, "\mathsf{T}_B\ \mathsf{map}_{\mathsf{T}}\ f"']
\end{tikzcd}$$

Proof by coinduction

In general one also gets:

$$\begin{aligned} \mathsf{map}_\mathsf{T}\, \mathsf{id}_A &= \mathsf{id}_{\mathsf{map}_{\mathsf{T}_A}} \\ \mathsf{map}_\mathsf{T}\, f \cdot [\![(p)]\!]_\mathsf{T} &= [\![(\mathsf{T}\,(f, \mathsf{id}) \cdot p)]\!]_\mathsf{T} \end{aligned}$$

- function map extends to a functor mapping a set A into the behaviour space of T-coalgebras parametric on A
- the last equation acts as an absorption law for coinductive extension

Proof by coinduction

Example: Lambek's Lemma
The dynamics of the final coalgebra is an isomorphism

proof idea:

- Assume the existence of an inverse α_{T} to $\omega_{\mathsf{T}} : \mathsf{T}\nu_{\mathsf{T}} \longleftarrow \nu_{\mathsf{T}}$.
  Then, $\alpha_{\mathsf{T}} \cdot \omega_{\mathsf{T}} = \mathsf{id}_{\nu_{\mathsf{T}}}$ and $\omega_{\mathsf{T}} \cdot \alpha_{\mathsf{T}} = \mathsf{id}_{\mathsf{T}\nu_{\mathsf{T}}}$
- Take one of this requirements and use it to conjecture a definition for α_{T} (or an implementation ...)
  Note the use of the reflection law to introduce an anamorphism in the calculation, instead of eliminating one
- Then check the validity of this conjecture by verifying with it the other requirement

Proof by coinduction

$$
\begin{array}{ll}
& \alpha_{\mathsf{T}} \cdot \omega_{\mathsf{T}} = \mathrm{id}_{\nu_{\mathsf{T}}} \\
\Leftrightarrow & \quad \{ \text{ reflection law } \} \\
& \alpha_{\mathsf{T}} \cdot \omega_{\mathsf{T}} = [\![\omega_{\mathsf{T}}]\!] \\
\Leftrightarrow & \quad \{ \text{ universal law } \} \\
& \omega_{\mathsf{T}} \cdot \alpha_{\mathsf{T}} \cdot \omega_{\mathsf{T}} = \mathsf{T}(\alpha_{\mathsf{T}} \cdot \omega_{\mathsf{T}}) \cdot \omega_{\mathsf{T}} \\
\Leftrightarrow & \quad \{ \text{ as a functor T preserves composition } \} \\
& \omega_{\mathsf{T}} \cdot \alpha_{\mathsf{T}} \cdot \omega_{\mathsf{T}} = \mathsf{T}\alpha_{\mathsf{T}} \cdot \mathsf{T}\omega_{\mathsf{T}} \cdot \omega_{\mathsf{T}} \\
\Leftrightarrow & \quad \{ \text{ cancel } \omega_{\mathsf{T}} \text{ from both sides \& universal law } \} \\
& \alpha_{\mathsf{T}} = [\![\mathsf{T}\omega_{\mathsf{T}}]\!]
\end{array}
$$

Proof by coinduction

$$
\begin{array}{ll}
 & \omega_{\mathsf{T}} \cdot \alpha_{\mathsf{T}} \\
= & \qquad \{ \text{ replace } \alpha_{\mathsf{T}} \text{ by the derived conjecture } \} \\
 & \omega_{\mathsf{T}} \cdot [\![(\mathsf{T}\omega_{\mathsf{T}})]\!] \\
= & \qquad \{ \ [\![(\mathsf{T}\omega_{\mathsf{T}})]\!] \text{ is a morphism } \} \\
 & \mathsf{T}[\![(\mathsf{T}\omega_{\mathsf{T}})]\!] \cdot \mathsf{T}\omega_{\mathsf{T}} \\
= & \qquad \{ \text{ as a functor } \mathsf{T} \text{ preserves composition } \} \\
 & \mathsf{T}([\![(\mathsf{T}\omega_{\mathsf{T}})]\!] \cdot \omega_{\mathsf{T}}) \\
= & \qquad \{ \text{ just proved } \} \\
 & \mathsf{T}\,\mathsf{id}_{\nu_{\mathsf{T}}} \\
= & \qquad \{ \text{ as a functor } \mathsf{T} \text{ preserves identities } \} \\
 & \mathsf{id}_{(\mathsf{T}\mathsf{id}_{\nu_{\mathsf{T}}})}
\end{array}
$$

Question

The powerset functor has not a final coalgebra. Why?

- Moore and Mealy transducers
- Behavioural effects
- The general case: coalgebras
- Bisimulation and bisimilarity

Bisimulation

A bisimulation is a relation over the state spaces of two coalgebras, p and q, which is closed for their dynamics, i.e.

$$(x, y) \in R \Rightarrow (p\,x, q\,y) \in \mathsf{T} R$$

which is PF-transformed to

$$R \subseteq p^{\circ} \cdot (\mathsf{T} R) \cdot q$$

Shunting on p° yields

$$p \cdot R \subseteq (\mathsf{T} R) \cdot q$$

Note: signature functor T is now extended to a relator.

Example: $\mathsf{T}\,X \;=\; B \times X$

$$
\begin{tikzcd}
U \arrow[d, "p"] & V \arrow[l, "R"'] \arrow[d, "q"] \\
B \times U & B \times V \arrow[l, "\mathrm{id} \times R"']
\end{tikzcd}
$$

$$
\begin{array}{ll}
 & p \cdot R \subseteq (\mathrm{id} \times R) \cdot q \\
\Leftrightarrow & \qquad \{ \text{ shunting } \} \\
 & R \subseteq p^{\circ} \cdot (\mathrm{id} \times R) \cdot q \\
\Leftrightarrow & \qquad \{ \text{ introducing variables } \} \\
 & \langle \forall\, u, v \;:\; u \in U, v \in V :\; u\,R\,v \;\Rightarrow\; u\,(p^{\circ} \cdot (\mathrm{id} \times R) \cdot q)\,v \rangle \\
\Leftrightarrow & \qquad \{ \text{ "guardanapo" rule } \} \\
 & \langle \forall\, u, v \;:\; u \in U, v \in V :\; u\,R\,v \;\Rightarrow\; p\,u\,(\mathrm{id} \times R)\,q\,v \rangle \\
\Leftrightarrow & \qquad \{ \text{ product } \} \\
 & \langle \forall\, u, v \;:\; u \in U, v \in V :\; u\,R\,v \;\Rightarrow\; \pi_1(p\,u) = \pi_1(q\,v) \;\wedge\; \pi_2(p\,u)\,R\,\pi_2(q\,v) \rangle
\end{array}
$$

Example: $\mathsf{T}\,X = \mathcal{P}X$

- Note that every powerset coalgebra can be regarded as the transpose of a binary relation through isomorphism

$$f = \Lambda R \quad \Leftrightarrow \quad R = \in \cdot f \tag{4}$$

- The powerset relator is defined by

$$\mathcal{P}R \quad = \quad (\in \backslash (R \cdot \in)) \cap (\in \backslash (R^\circ \cdot \in))^\circ \tag{5}$$

where $\cap$ denotes relation intersection and $R \setminus S$ denotes relational division,

$$a(R \setminus S)c \;\Leftrightarrow\; \langle \forall\, b \;:\; b\, R\, a :\; b\, S\, c \rangle$$

a relational operator whose semantics is captured by universal property

$$R \cdot X \subseteq S \quad \Leftrightarrow \quad X \subseteq R \setminus S \tag{6}$$

Then,

Example: $\mathsf{T}\,X = \mathcal{P}X$

$$
\begin{array}{ll}
& p \cdot R \subseteq (\mathcal{P}R) \cdot q \\
\Leftrightarrow & \quad \{ \text{ let } p, q := \Lambda P, \Lambda Q, \text{ unfold } \mathcal{P}R \text{ (5) } \} \\
& (\Lambda P) \cdot R \subseteq (\in \backslash (R \cdot \in)) \cap (\in \backslash (R^{\circ} \cdot \in))^{\circ} \cdot (\Lambda Q) \\
\Leftrightarrow & \quad \{ \text{ distribution (since } \Lambda Q \text{ is a function) } \} \\
& (\Lambda P) \cdot R \subseteq (\in \backslash (R \cdot \in)) \cdot (\Lambda Q) \ \wedge \ (\Lambda P) \cdot R \subseteq (\in \backslash (R^{\circ} \cdot \in))^{\circ} \cdot (\Lambda Q) \\
\Leftrightarrow & \quad \{ \text{ property } R \backslash (S \cdot f) = (R \backslash S) \cdot f \text{ ; converses } \} \\
& (\Lambda P) \cdot R \subseteq \in \backslash (R \cdot \in \cdot \Lambda Q) \ \wedge \ R^{\circ} \cdot (\Lambda P)^{\circ} \subseteq (\Lambda Q)^{\circ} \cdot (\in \backslash (R^{\circ} \cdot \in)) \\
\Leftrightarrow & \quad \{ \text{ shunting and property above } \} \\
& (\Lambda P) \cdot R \subseteq \in \backslash (R \cdot \in \cdot \Lambda Q) \ \wedge \ (\Lambda Q) \cdot R^{\circ} \subseteq \in \backslash (R^{\circ} \cdot \in \cdot \Lambda P) \\
\Leftrightarrow & \quad \{ \text{ (6) twice } \} \\
& \in \cdot (\Lambda P) \cdot R \subseteq R \cdot \in \cdot \Lambda Q \ \wedge \ \in \cdot (\Lambda Q) \cdot R^{\circ} \subseteq R^{\circ} \cdot \in \cdot \Lambda P \\
\Leftrightarrow & \quad \{ \text{ cancellation } \in \cdot (\Lambda R) = R \text{ four times } \} \\
& P \cdot R \subseteq R \cdot Q \ \wedge \ Q \cdot R^{\circ} \subseteq R^{\circ} \cdot P
\end{array}
$$

Example: $\mathsf{T}X = \mathcal{P}X$

The two conjuncts state that R and its converse are simulations between state transition relations P and Q, which corresponds to the Park-Milner definition:

- a bisimulation is a simulation such that its converse is also a simulation
- a simulation between relations P and Q is a relation R such that, if $(p, q) \in R$, then for all p' such that $(p', p) \in P$, then there is a q' such that $(p', q') \in R$ and $(q', q) \in Q$

because

Example: $\mathsf{T}\,X = \mathcal{P}X$

$$
\begin{array}{ll}
 & P \cdot R \subseteq R \cdot Q \\
\Leftrightarrow & \quad \{ \; S\cdot \vdash R\backslash \; \} \\
 & R \subseteq P\backslash(R \cdot Q) \\
\Leftrightarrow & \quad \{ \text{ PF transform } \} \\
 & \langle \forall\ u, v \ :\ u \in U \wedge v \in V :\ uRv \Rightarrow u(P\backslash(R \cdot Q))v \rangle \\
\Leftrightarrow & \quad \{ \text{ definition of right division } \} \\
 & \langle \forall\ u, v \ :\ u \in U \wedge v \in V :\ uRv \Rightarrow \langle \forall\ u' \ :\ u' \in U \wedge u'Pu :\ u'(R \cdot Q)v \rangle \rangle \\
\Leftrightarrow & \quad \{ \text{ quantifier trading, nesting and trading again } \} \\
 & \langle \forall\ u, u', v \ :\ u, u' \in U \wedge v \in V :\ uRv \wedge u'Pu \Rightarrow u'(R \cdot Q)v \rangle \\
\Leftrightarrow & \quad \{ \text{ relational composition } \} \\
 & \langle \forall\ u, u', v \ :\ u, u' \in U \wedge v \in V :\ uRv \wedge u'Pu \Rightarrow \langle \exists\ v' \ :\ v' \in V :\ u'Rv' \wedge v'Qv \rangle \rangle
\end{array}
$$

Example: $\mathsf{T}\,X = \mathcal{P}X$

and

$$
\begin{array}{ll}
 & Q \cdot R^{\circ} \subseteq R^{\circ} \cdot P \\
\Leftrightarrow & \qquad \{ \text{ by a similar argument } \} \\
 & \langle \forall\ u, v, v' \ :\ u \in U \wedge v, v' \in V :\ uRv \wedge v'Qv \Rightarrow \langle \exists\ u' \ :\ u' \in U :\ u'Rv' \wedge u'Pu \rangle \rangle
\end{array}
$$

which jointly states that both R and R° are simulations

Example: $\mathsf{T}\,X = \mathcal{P}(B \times X)$

This result scales easily for $\mathsf{T}\,X = \mathcal{P}(B \times X)$ coalgebras, where it is usually expressed in terms of B-indexed families of transition relations:

- R is a simulation between coalgebras p and q as before iff, for all $b \in B$, $u, u' \in U$ and $v \in V$,

$$uRv \wedge u' \xrightarrow{b}_p u \Rightarrow \langle \exists\, v' \;:\; v' \in V :\; u'Rv' \wedge v' \xrightarrow{b}_q v \rangle$$

- R is a bisimulation iff both R and R° are simulations

which leads to the usual definition of bisimulation in process algebra (cf, [Milner, 80])

Example: $\mathrm{T}\,X = \mathcal{P}(B \times X)$

$\boxed{\text{Example}}$ states q_0 and p_0 in coalgebras

$$
\begin{tikzcd}
 & q_1 \arrow[r, "d"] & q_2 \\
q_0 \arrow[ur, "a"] \arrow[dr, "a"'] & & \\
 & q_4 \arrow[r, "e"'] & q_3
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & & p_2 \\
p_0 \arrow[r, "a"] & p_1 \arrow[ur, "d"] \arrow[dr, "e"'] & \\
 & & p_3
\end{tikzcd}
$$

are related by simulation

$$\{\langle q_0, p_0\rangle, \langle q_1, p_1\rangle, \langle q_4, p_1\rangle, \langle q_2, p_2\rangle, \langle q_3, p_3\rangle\}$$

Example: $\mathrm{T}\,X = \mathcal{P}(B \times X)$

Note, however, that, although there are simulations R and S containing pairs (q_0, p_0) and (p_0, q_0) in

$$
\begin{tikzcd}
 & q_1 & \\
q_0 \arrow[ur, "a"] \arrow[dr, "a"'] & & & p_0 \arrow[r, "a"] & p_1 \arrow[r, "b"] & p_3 \\
 & q_2 \arrow[r, "b"] & q_3
\end{tikzcd}
$$

the two states are not bisimilar.

Exercise. Compute relations R and S above and explain why q_0 and p_0 are not bisimilar.

Exercise. Compute the definition of bisimulation for the signature functor of a Moore and a Mealy transducer, respectively.

Bisimulation as a Reynolds arrow

The definition of bisimulation brings to mind the "Reynolds arrow combinator"-pattern:

$$f(R \leftarrow S)g \quad \Leftrightarrow \quad f \cdot S \subseteq R \cdot g$$

leading to

$$R \text{ is a bisimulation} \quad \Leftrightarrow \quad p(\mathsf{T}R \leftarrow R)q \tag{7}$$

Note: Reynolds' arrow combinator is a relation on functions useful in expressing properties of functions, notably the "free theorem" of a polymorphic function f:

$$\mathsf{G}A \xleftarrow{f} \mathsf{T}A \text{ polymorphic} \quad \Leftrightarrow \quad \langle \forall\ R\ ::\ f(\mathsf{G}R \leftarrow \mathsf{T}R)f \rangle$$

Reynolds-arrow laws

$$
\begin{array}{rcll}
id \leftarrow id & = & id & (8) \\
(R \leftarrow S)^{\circ} & = & R^{\circ} \leftarrow S^{\circ} & (9) \\
(R \leftarrow V) \cdot (S \leftarrow U) & \subseteq & (R \cdot S) \leftarrow (V \cdot U) & (10) \\
R \leftarrow S \quad \subseteq \quad V \leftarrow U & \Leftarrow & R \subseteq V \wedge U \subseteq S & (11) \\
k(f \leftarrow g)h & \Leftrightarrow & k \cdot g = f \cdot h & (12) \\
(f \leftarrow g^{\circ})h & = & f \cdot h \cdot g & (13)
\end{array}
$$

Reynolds-arrow laws

Property (11) entails monotonicity on the left hand side, thus,

$$S \leftarrow R \subseteq (S \cup V) \leftarrow R \tag{14}$$

$$\top \leftarrow S = \top \tag{15}$$

and anti-monotonicity on the right hand side:

$$R \leftarrow \bot = \top \tag{16}$$

as well as two distributive properties:

$$S \leftarrow (R_1 \cup R_2) = (S \leftarrow R_1) \cap (S \leftarrow R_2) \tag{17}$$

$$(S_1 \cap S_2) \leftarrow R = (S_1 \leftarrow R) \cap (S_2 \leftarrow R) \tag{18}$$

Bisimulation: Properties

- The converse of a bisimulation is also a bisimulation

$$
\begin{array}{ll}
 & R \text{ is a bisimulation} \\
\Leftrightarrow & \quad \{ \ (7) \ \} \\
 & p(\mathsf{T}R \leftarrow R)q \\
\Leftrightarrow & \quad \{ \ \text{converse} \ \} \\
 & q(\mathsf{T}R \leftarrow R)^\circ p \\
\Leftrightarrow & \quad \{ \ (9) \ \} \\
 & p((\mathsf{T}R)^\circ \leftarrow R^\circ)q \\
\Leftrightarrow & \quad \{ \ \text{relator } \mathsf{T} \ \} \\
 & q(\mathsf{T}(R^\circ) \leftarrow R^\circ)p \\
\Leftrightarrow & \quad \{ \ (7) \ \} \\
 & R^\circ \text{ is a bisimulation}
\end{array}
$$

- Composition of bisimulations is a bisimulation
by property (10), as can be checked by parsing its pointwise version: for all suitably typed coalgebras p and q,

$$\langle \exists\ z \ :: \ p(\mathsf{T}S \leftarrow S)z \wedge z(\mathsf{T}R \leftarrow R)q \rangle \quad \Rightarrow \quad p(\mathsf{T}(S \cdot R) \leftarrow (S \cdot R))q$$

Bisimulation: Properties

- the identity relation *id* is a bisimulation

$$
\begin{array}{ll}
 & p(\mathsf{T}\,id \leftarrow id)q \\
\Leftrightarrow & \quad \{ \text{ relator } \mathsf{T} \ \} \\
 & p(id \leftarrow id)q \\
\Leftrightarrow & \quad \{ \ (8) \ \} \\
 & p = q
\end{array}
$$

- the empty relation $\perp$ is a bisimulation

$$
\begin{array}{ll}
 & \langle \forall\ p, q \ :: \ p(\mathsf{T}\perp \leftarrow \perp)q \rangle \\
\Leftrightarrow & \quad \{ \text{ PF-transform } \} \\
 & \langle \forall\ p, q \ :: \ p(\mathsf{T}\perp \leftarrow \perp)q \Leftrightarrow \textsc{True} \rangle \\
\Leftrightarrow & \quad \{ \text{ PF-transform } \} \\
 & \mathsf{T}\perp \leftarrow \perp = \top \\
\Leftrightarrow & \quad \{ \ (16) \ \} \\
 & \textsc{True}
\end{array}
$$

Bisimulation: Properties

- bisimulations are closed under union

$$p(\mathsf{T}R_1 \leftarrow R_1)q \wedge p(\mathsf{T}R_2 \leftarrow R_2)q \quad \Rightarrow \quad p(\mathsf{T}(R_1 \cup R_2) \leftarrow (R_1 \cup R_2))q \quad (19)$$

stems from properties (11,14) and (17). First we PF-transform (19) to

$$(\mathsf{T}R_1 \leftarrow R_1) \cap (\mathsf{T}R_2 \leftarrow R_2) \quad \subseteq \quad \mathsf{T}(R_1 \cup R_2) \leftarrow (R_1 \cup R_2)$$

and reason:

$$\begin{array}{cl} & (\mathsf{T}R_1 \leftarrow R_1) \cap (\mathsf{T}R_2 \leftarrow R_2) \\ \subseteq & \quad \{ \ (14)\ (\text{twice})\ ;\ \text{monotonicity of } \cap \ \} \\ & ((\mathsf{T}R_1 \cup \mathsf{T}R_2) \leftarrow R_1) \cap ((\mathsf{T}R_1 \cup \mathsf{T}R_2) \leftarrow R_2) \\ = & \quad \{ \ (17) \ \} \\ & (\mathsf{T}R_1 \cup \mathsf{T}R_2) \leftarrow (R_1 \cup R_2) \\ \subseteq & \quad \{ \ \mathsf{F} \text{ is monotonic; } (11) \ \} \\ & \mathsf{T}(R_1 \cup R_2) \leftarrow (R_1 \cup R_2) \end{array}$$

Bisimulation: Properties

- any coalgebra morphism is a bisimulation (why?)
- behavioural equivalence is a bisimulation.

$$
\begin{array}{ll}
 & p(\mathsf{T}(\llbracket p \rrbracket^{\circ} \cdot \llbracket q \rrbracket) \leftarrow \llbracket p \rrbracket^{\circ} \cdot \llbracket q \rrbracket)q \\
\Leftrightarrow & \qquad \{ \text{ definition } \} \\
 & \llbracket p \rrbracket^{\circ} \cdot \llbracket q \rrbracket \subseteq p^{\circ} \cdot \mathsf{T}(\llbracket p \rrbracket^{\circ} \cdot \llbracket q \rrbracket) \cdot q \\
\Leftrightarrow & \qquad \{ \text{ relators } \} \\
 & \llbracket p \rrbracket^{\circ} \cdot \llbracket q \rrbracket \subseteq p^{\circ} \cdot \mathsf{T}\llbracket p \rrbracket^{\circ} \cdot \mathsf{T}\llbracket q \rrbracket \cdot q \\
\Leftrightarrow & \qquad \{ \text{ converse } \} \\
 & \llbracket p \rrbracket^{\circ} \cdot \llbracket q \rrbracket \subseteq (\mathsf{T}\llbracket p \rrbracket \cdot p)^{\circ} \cdot \mathsf{T}\llbracket q \rrbracket \cdot q \\
\Leftrightarrow & \qquad \{ \text{ universal property of coinductive extension } \} \\
 & \llbracket p \rrbracket^{\circ} \cdot \llbracket q \rrbracket \subseteq (\omega \cdot \llbracket p \rrbracket)^{\circ} \cdot \omega \cdot \llbracket q \rrbracket \\
\Leftrightarrow & \qquad \{ \text{ converse } \} \\
 & \llbracket p \rrbracket^{\circ} \cdot \llbracket q \rrbracket \subseteq \llbracket p \rrbracket^{\circ} \cdot \omega^{\circ} \cdot \omega \cdot \llbracket q \rrbracket \\
\Leftrightarrow & \qquad \{ \text{ Lambek (final coalgebra is an isomorphism) } \} \\
 & \textsc{True}
\end{array}
$$

Bisimilarity

Def. Two states, u and v, from the same or different coalgebras, are bisimilar iff they are related by a bisimulation, i.e.,

$$u \sim v \;\Leftrightarrow\; \langle \exists\, R \;:\; R \subseteq U \times V \;:\; uRv \wedge R \text{ is a bisimulation} \rangle$$

Th. Bisimilarity is an equivalence relation.

Th. The set of all bisimulations, defined between two coalgebras, over state spaces U and V, is a complete lattice, ordered by $\subseteq$, whose top is the restriction of $\sim$ to $U \times V$.

Exercise. Prove both theorems.