Coalgebraic Structures in Program Construction

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Motivation

How to drive software production into solid engineering standards?

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An old recipe from school physics:

- Understand the problem
- Create a **mathematical model**
- **Reason** within the model
- Calculate a **solution** ($\equiv$ an **implementation**)

The Approach

- combine previous experience with model oriented formal design methods (VDM and CAMILA [AMAST'97])
- ... with the 'tradition' of program calculi driven by (initial or final) type specifications
  - *cf.*, systematic derivation of algorithms in a way that correctness is guaranteed by construction
  - 'twin' concern in functional programming [McCarthy original papers, Backus FP]
  - Bird-Meertens 'school' [BM87] of mathematics of program construction

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Key: resort to the algebra *vs* coalgebra duality as a mathematical 'explanation' of the intuitive symmetry between data and behavioural structures

- ... a 'big' topic: from 'non well-founded' sets [Aczel88]
- to 'behavioural satisfaction' [Reichel81] and 'final semantics' [Plotkin, Rutten, Turi 93]
- to axiomatic 'coalgebraic specification [Jacobs97] and 'coalgebraic logic' [Moss99]
- ... partly documented in the CMCS series (from 1998)

The Basic Symmetry

(Initial) algebras are abstract description of data structures

$$[\mathsf{nil}, \mathsf{cons}] : \mathbf{1} + O \times L \longrightarrow L$$

In general:

a tool box: 

an assembly process:  artifact $\xrightarrow{d}$ artifact

The emphasis is on construction

The Basic Symmetry

(Final) coalgebras are abstract descriptions of systems' behaviours

$$\langle \mathsf{at}, \mathsf{m} \rangle : U \longrightarrow O \times U$$

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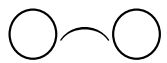
In general:

a lens: 

an observation structure: universe $\xrightarrow{f}$ universe

The emphasis is on observation

The Observational Viewpoint

Dynamical Systems

- internal state space (‘memory’ and persistence)
- possibility of interaction with other components during the overall computation
- observable through well-defined interfaces to ensure flow of data

found everywhere: from sophisticated plant control systems to formal automata or domestic appliances.

The Observational Viewpoint

Observation Shapes

- ‘opaque’

- black & white

- colouring

... in each case the colour set acts as a classifier of the state space

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Observation Shapes

- partiality

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- external stimulus

$$\bigcirc\!\!\frown\!\!\bigcirc\, U \;=\; U^{I}$$

- non determinism

$$\bigcirc\!\!\frown\!\!\bigcirc\, U \;=\; \mathcal{P}U$$

A parenthesis

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space of types and typed arrows ...

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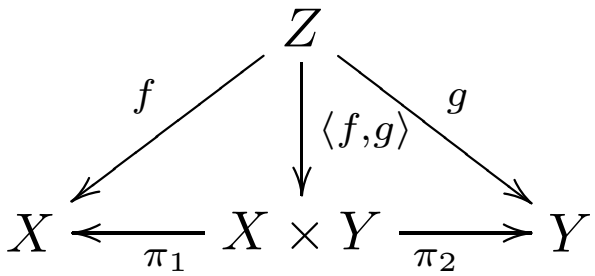
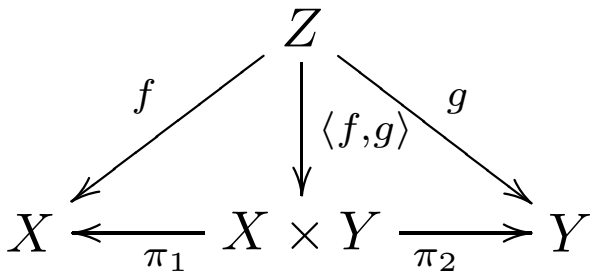
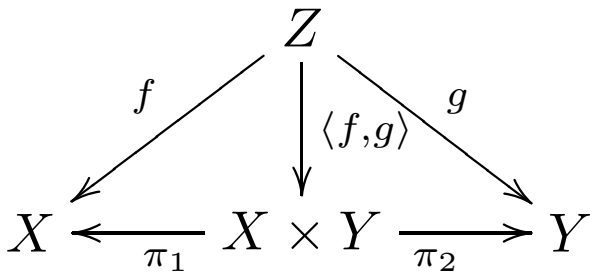
Example: pointfree presentation of the product construction


Z
f
⟨f,g⟩
g
X
π1
X × Y
π2
Y



Universal property:

$$k = \langle f, g \rangle \quad \Leftrightarrow \quad \pi_1 \cdot k = f \ \wedge \ \pi_2 \cdot k = g$$



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Laws:

$$\pi_1 \cdot \langle f, g \rangle = f \ , \pi_2 \cdot \langle f, g \rangle = g$$

$$\langle \pi_1, \pi_2 \rangle \ = \ \mathsf{id}_{X \times Y}$$

$$\langle g, h \rangle \cdot f \ = \ \langle g \cdot f, h \cdot f \rangle$$

$$(i \times j) \cdot \langle g, h \rangle \ = \ \langle i \cdot g, j \cdot h \rangle$$

End of parenthesis

)

Systems & Behaviours

Simple Objects

$$p = \langle \mathsf{at}, \mathsf{m} \rangle : U \longrightarrow O \times U$$

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$$p \;=\; \langle \mathsf{at}, \mathsf{m} \rangle : U \longrightarrow O \times U$$

The behaviour of p at (from) a state u is revealed by successive observations (experiments):

$$[\![(p)]\!]\; u \;=\; <\mathsf{at}\; u,\; \mathsf{at}\; (\mathsf{m}\; u),\; \mathsf{at}\; (\mathsf{m}\; (\mathsf{m}\; u)), ...>$$

Systems & Behaviours

Mealy Machines

$$p = \overline{\langle \mathsf{at}, \mathsf{m} \rangle} : U \longrightarrow (O \times U)^I$$

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Behaviours are elements of O^{I^+} — *i.e.*, functions from non empty sequences of I to O

$$[\![(p)]\!]\; u \;=\; \lambda <s : i> \;.\;\; \mathsf{at}\; \langle (\mathsf{next}\; u)\; s, i \rangle$$

where

$$(\mathsf{next}\; u) <> = u \qquad \text{and} \qquad (\mathsf{next}\; u) <s : i> = \mathsf{m}\; \langle (\mathsf{next}\; u)\; s, i \rangle$$

Systems & Behaviours

Fact: The behaviours of ◯⌒◯-systems form a ◯⌒◯-system

$$\omega \;=\; \overline{\langle \mathsf{root}, \mathsf{branches} \rangle} : O^{I^+} \longrightarrow (O \times O^{I^+})^I$$

where,

$$\mathsf{root}\ \langle \phi, i \rangle = \phi\ {<}i{>} \qquad \text{and} \qquad \mathsf{branches}\ \langle \phi, i \rangle = \lambda\, s\ .\ \ \phi\ {<}i : s{>}$$

Systems & Behaviours

Relating Systems

A morphism is a function connecting the state spaces which preserves the dynamics, *i.e.*,

$$\begin{array}{ccc} U \times I & \xrightarrow{\langle \mathsf{at},\mathsf{m} \rangle} & O \times U \\ {\scriptstyle h \times \mathsf{id}}\downarrow & & \downarrow{\scriptstyle \mathsf{id} \times h} \\ V \times I & \xrightarrow{\langle \mathsf{at}',\mathsf{m}' \rangle} & O \times V \end{array}$$

I.e.,

$$\mathsf{at} = \mathsf{at}' \cdot (h \times \mathsf{id})$$

and

$$h \cdot \mathsf{m} = \mathsf{m}' \cdot (h \times \mathsf{id})$$

Systems & Behaviours

Fact: $[\![(p)]\!]$ is a morphism from p to ω

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Proof (check both conditions)

$$
\begin{array}{ll}
 & (\mathsf{root} \cdot ([\![(p)]\!] \times \mathsf{id}))\ \langle u, i\rangle \\
= & \quad \{\ \mathsf{root}\ \text{defi nition}\ \} \\
 & ([\![(p)]\!]\ u)\ <i> \\
= & \quad \{\ [\![(p)]\!]\ \text{defi nition}\ \} \\
 & \mathsf{at}\ \langle (\mathsf{next}\ u)\ <>, i\rangle \\
= & \quad \{\ \mathsf{next}\ \text{defi nition}\ \} \\
 & \mathsf{at}\ \langle u, i\rangle
\end{array}
$$

Systems & Behaviours

and

$$
\begin{aligned}
& (\mathsf{branches} \cdot ([\![p]\!] \times \mathsf{id}))\ \langle u, i\rangle \\
= & \quad \{\ \mathsf{branches}\ \text{defi nition}\ \} \\
& \lambda <s : j> .\ ([\![p]\!]\ u) <i : s : j> \\
= & \quad \{\ [\![p]\!]\ \text{defi nition}\ \} \\
& \lambda <s : j> .\ \mathsf{at}\ \langle (\mathsf{next}\ u) <i : s>, j\rangle \\
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& \lambda <s : j> .\ \mathsf{at}\ \langle (\mathsf{next\ m}\ \langle u, i\rangle)s, j\rangle \\
= & \quad \{\ [\![p]\!]\ \text{defi nition}\ \} \\
& [\![p]\!]\ (\mathsf{m}\ \langle u, i\rangle)
\end{aligned}
$$

Systems & Behaviours

Fact: Morphisms preserve behaviour

$$[\![(p)]\!]\ u \ = \ [\![(p)]\!]\ h\ u$$

Systems & Behaviours

What's special about $\langle O^{I^+}, \omega \rangle$?

- There is always a morphism — $[\![(p)]\!]$ — to it from any $\langle U, p = \overline{\langle \mathsf{at}, \mathsf{m} \rangle} \rangle$

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What's special about $\langle O^{I^+}, \omega \rangle$?

- There is always a morphism — $[\![(p)]\!]$ — to it from any $\langle U, p = \overline{\langle \mathsf{at}, \mathsf{m} \rangle} \rangle$
- Because morphisms preserve behaviour, such a morphism is unique
  - $\omega = \langle \mathsf{root}, \mathsf{branches} \rangle$ is the final Mealy machine
  - example of a characterisation by a universal property

Going Generic

- Mealy machines (on I and O) were introduced as systems observed through

$$\bigcirc\!\!\frown\!\!\bigcirc \;=\; (O \times -)^I$$

$\bigcirc\!\!\frown\!\!\bigcirc$ as a mapping to decompose the ‘observable-universe’ U into an ’observation context’ $(O \times U)^I$

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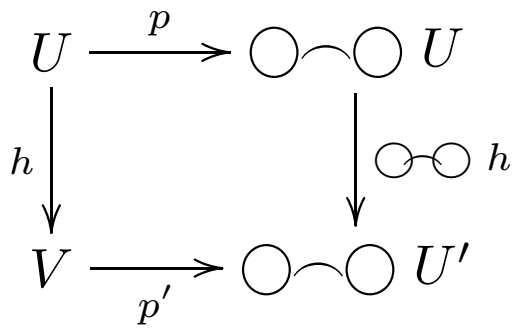
- 

Going Generic

Interfaces are Functors

A functor is a function over our working universe of typed functions, which which preserves identities and composition

$$\begin{aligned} \mathsf{T}\mathrm{id}_X &= \mathrm{id}_{\mathsf{T}X} \\ \mathsf{T}(f \cdot g) &= \mathsf{T}f \cdot \mathsf{T}g \end{aligned}$$

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... i.e., the (partial) monoid structure

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Final Coalgebras and Behaviours

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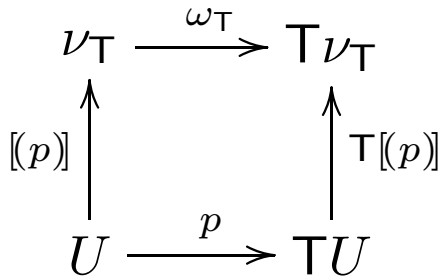
- Any morphism between two T-coalgebras preserves behaviour;
- Whenever the space of behaviours of a class of T-coalgebras can be turned on a T-coalgebra itself

$$\omega_{\mathsf{T}} : \nu_{\mathsf{T}} \longrightarrow \mathsf{T}\nu_{\mathsf{T}}$$

this is the final coalgebra: from any other T-coalgebra p there is a unique morphism $[\![(p)]\!]$ making the following diagram to commute:



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The universal property is equivalently captured by the following law:

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Example: Lambek's Lemma

The dynamics of the final coalgebra is an isomorphism

proof idea:

- Assume the existence of an inverse α_{T} to $\omega_{\mathsf{T}} : \nu_{\mathsf{T}} \longrightarrow \mathsf{T}\nu_{\mathsf{T}}$.
  Then, $\alpha_{\mathsf{T}} \cdot \omega_{\mathsf{T}} = \mathrm{id}_{\nu_{\mathsf{T}}}$ and $\omega_{\mathsf{T}} \cdot \alpha_{\mathsf{T}} = \mathrm{id}_{\mathsf{T}\nu_{\mathsf{T}}}$

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- Take one of this requirements and use it to conjecture a definition for α_{T} (or an implementation ...)
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- Then check the validity of this conjecture by verifying with it the other requirement

Example: the conjecture step

$$\alpha_{\mathsf{T}} \cdot \omega_{\mathsf{T}} = \mathsf{id}_{\nu_{\mathsf{T}}}$$

$=$ { reflection law }

$$\alpha_{\mathsf{T}} \cdot \omega_{\mathsf{T}} = [\!(\omega_{\mathsf{T}})\!]$$

$=$ { anamorphism universal characterisation }

$$\omega_{\mathsf{T}} \cdot \alpha_{\mathsf{T}} \cdot \omega_{\mathsf{T}} = \mathsf{T}(\alpha_{\mathsf{T}} \cdot \omega_{\mathsf{T}}) \cdot \omega_{\mathsf{T}}$$

$=$ { as a functor T preserves composition }

$$\omega_{\mathsf{T}} \cdot \alpha_{\mathsf{T}} \cdot \omega_{\mathsf{T}} = \mathsf{T}\alpha_{\mathsf{T}} \cdot \mathsf{T}\omega_{\mathsf{T}} \cdot \omega_{\mathsf{T}}$$

$=$ { cancel ω_{T} from both sides & ana universal}

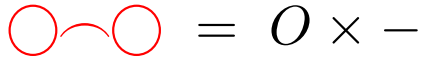
$$\alpha_{\mathsf{T}} = [\!(\mathsf{T}\omega_{\mathsf{T}})\!]$$

Example: the verification step

$$
\begin{array}{ll}
 & \omega_{\mathsf{T}} \cdot \alpha_{\mathsf{T}} \\
= & \quad \{ \text{ replace } \alpha_{\mathsf{T}} \text{ by the derived conjecture } \} \\
 & \omega_{\mathsf{T}} \cdot [\!(\mathsf{T}\omega_{\mathsf{T}})\!] \\
= & \quad \{ \ [\!(\mathsf{T}\omega_{\mathsf{T}})\!] \text{ is a morphism } \} \\
 & \mathsf{T}[\!(\mathsf{T}\omega_{\mathsf{T}})\!] \cdot \mathsf{T}\omega_{\mathsf{T}} \\
= & \quad \{ \text{ as a functor T preserves composition } \} \\
 & \mathsf{T}([\!(\mathsf{T}\omega_{\mathsf{T}})\!] \cdot \omega_{\mathsf{T}}) \\
= & \quad \{ \text{ just proved } \} \\
 & \mathsf{T}\ \mathsf{id}_{\nu_{\mathsf{T}}} \\
= & \quad \{ \text{ as a functor T preserves identities } \} \\
 & \mathsf{id}_{(\mathsf{T}\mathsf{id}_{\nu_{\mathsf{T}}})}
\end{array}
$$

Example: Programming with Streams

Final system for



$$= O \times -$$

Example: Programming with Streams

Final system for

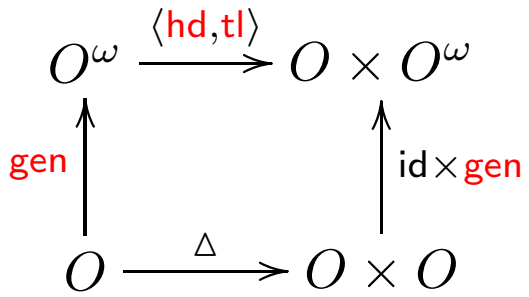
$$= O \times -$$

$$\langle O^\omega, \langle \mathsf{hd}, \mathsf{tl} \rangle : O^\omega \longrightarrow O \times O^\omega \rangle$$

Behaviours $\equiv$ coinductive types

Example: Programming with Streams

Stream Generation



$$\mathsf{gen} = [\![(\Delta)]\!]$$

Δ carries the 'genetic inheritance' of the generating process

From a programming viewpoint it is the eureka! step

Example: Programming with Streams

Coinductive Definition = behaviour specification under all the observers

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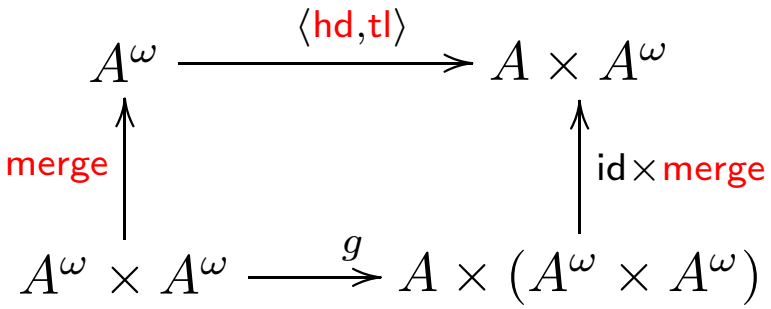
Coinductive Definition = behaviour specification under all the observers

$$
\begin{aligned}
& (\mathsf{id} \times \mathsf{gen}) \cdot \triangle = \langle \mathsf{hd}, \mathsf{tl} \rangle \cdot \mathsf{gen} \\
= & \quad \{ \ \triangle \text{ defi nition} \ \} \\
& (\mathsf{id} \times \mathsf{gen}) \cdot \langle \mathsf{id}, \mathsf{id} \rangle = \langle \mathsf{hd}, \mathsf{tl} \rangle \cdot \mathsf{gen} \\
= & \quad \{ \ \times \text{ absorption and fusion} \ \} \\
& \langle \mathsf{id}, \mathsf{gen} \rangle = \langle \mathsf{hd} \cdot \mathsf{gen}, \mathsf{tl} \cdot \mathsf{gen} \rangle \\
= & \quad \{ \ \text{structural equality} \ \} \\
& \mathsf{hd} \cdot \mathsf{gen} = \mathsf{id} \ \wedge \ \mathsf{tl} \cdot \mathsf{gen} = \mathsf{gen} \\
= & \quad \{ \ \text{going pointwise} \ \} \\
& \mathsf{hd} \ (\mathsf{gen} \ a) = a \ \wedge \ \mathsf{tl} \ (\mathsf{gen} \ a) = \mathsf{gen} \ a
\end{aligned}
$$

Example: Programming with Streams

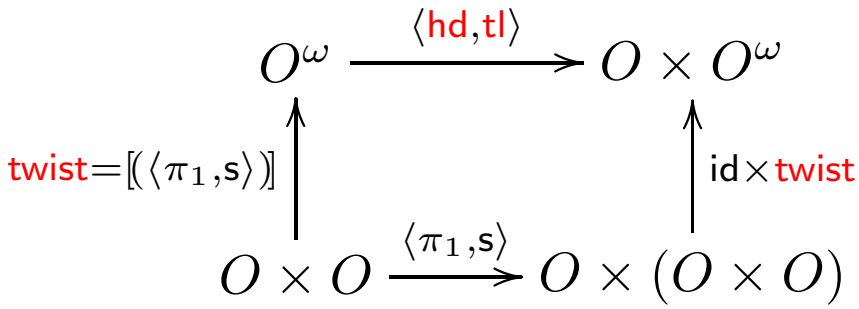
Stream Merge



$$g = \langle \mathsf{hd} \cdot \pi_1, \mathsf{s} \cdot (\mathsf{tl} \times \mathsf{id}) \rangle$$

Example: Programming with Streams

Twist



Example: Programming with Streams

$$\boxed{\mathsf{merge}\ (a^\omega, b^\omega) \;=\; (ab)^\omega}$$

i.e.

$$\mathsf{merge} \cdot (\mathsf{gen} \times \mathsf{gen}) \;=\; \mathsf{twist}$$

Example: Streams (Proof)

$$\mathsf{merge} \cdot (\mathsf{gen} \times \mathsf{gen}) \;=\; \mathsf{twist}$$

$=$ { merge definition }

$$[\![(\langle \mathsf{hd} \cdot \pi_1, \mathsf{s} \cdot (\mathsf{tl} \times \mathsf{id})\rangle)]\!] \cdot (\mathsf{gen} \times \mathsf{gen}) \;=\; [\![(\langle \pi_1, \mathsf{s}\rangle)]\!]$$

$\Leftarrow$ { anamorphism fusion }

$$\langle \mathsf{hd} \cdot \pi_1, \mathsf{s} \cdot (\mathsf{tl} \times \mathsf{id})\rangle \cdot (\mathsf{gen} \times \mathsf{gen}) \;=\; \mathsf{id} \times (\mathsf{gen} \times \mathsf{gen}) \cdot \langle \pi_1, \mathsf{s}\rangle$$

$=$ { $\times$ absorption and reflection }

$$\langle \mathsf{hd} \cdot \mathsf{gen} \cdot \pi_1, \mathsf{s} \cdot ((\mathsf{tl} \cdot \mathsf{gen}) \times \mathsf{gen})\rangle \;=\; \mathsf{id} \times (\mathsf{gen} \times \mathsf{gen}) \cdot \langle \pi_1, \mathsf{s}\rangle$$

$=$ { $\mathsf{tl} \cdot \mathsf{gen} = \mathsf{gen}$ and $\mathsf{hd} \cdot \mathsf{gen} = \mathsf{id}$ }

$$\langle \pi_1, \mathsf{s} \cdot (\mathsf{gen} \times \mathsf{gen})\rangle \;=\; \mathsf{id} \times (\mathsf{gen} \times \mathsf{gen}) \cdot \langle \pi_1, \mathsf{s}\rangle$$

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$$\langle \pi_1, \mathsf{s} \cdot (\mathsf{gen} \times \mathsf{gen}) \rangle = \langle \pi_1, (\mathsf{gen} \times \mathsf{gen}) \cdot \mathsf{s} \rangle$$

$=$ { s is natural, *i.e.*, $(f \times g) \cdot \mathsf{s} = \mathsf{s} \cdot (g \times f)$ }

$$\langle \pi_1, \mathsf{s} \cdot (\mathsf{gen} \times \mathsf{gen}) \rangle = \langle \pi_1, \mathsf{s} \cdot (\mathsf{gen} \times \mathsf{gen}) \rangle$$

Observational Equivalence

$$u \sim v \iff [\![(p)]\!]\, u = [\![(q)]\!]\, v$$

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Alternative: two states are equivalent if related by a morphism h (why?)
Then

$$\langle u, v\rangle \in R \;\Rightarrow\; u \sim v$$

where $R = \{\langle x, h\ x\rangle |\ x \in U\}$.

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Can this idea be generalised?
What properties must a relation R have
to conclude $u \sim v$ by checking if $\langle u, v\rangle$ is in R ?

Example: Streams

Given $p = \langle U, \langle \mathsf{at}_p, \mathsf{m}_p \rangle \rangle$ and $q = \langle V, \langle \mathsf{at}_q, \mathsf{m}_q \rangle \rangle$ and states u and v

when do they generate the same stream?

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- Check attributes: $\mathsf{at}_p\ u = \mathsf{at}_q\ v$,
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In other words, look for $R \subseteq U \times V$ such that

$$\langle u, v \rangle \in R \ \Rightarrow\ \mathsf{at}_p\ u = \mathsf{at}_q\ v\ \wedge\ \langle \mathsf{m}_p\ u, \mathsf{m}_q\ v \rangle \in R$$

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, R must be closed under the coalgebra dynamics — a bisimulation

Looking for Genericity

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Looking for Genericity

A simple calculation yields:

$$\mathsf{at}_\rho = \mathsf{at}_p \cdot \pi_1 = \mathsf{at}_q \cdot \pi_2 \ \wedge \ \mathsf{m}_\rho = \mathsf{m}_p \times \mathsf{m}_q$$

$$\equiv \quad \{ \ \times \text{ reflection} \}$$

$$\mathsf{at}_\rho = \mathsf{at}_p \cdot \pi_1 = \mathsf{at}_q \cdot \pi_2 \ \wedge \ \langle \pi_1, \pi_2 \rangle \cdot \mathsf{m}_\rho = \mathsf{m}_p \times \mathsf{m}_q$$

$$\equiv \quad \{ \ \times \text{ fusion and absorption} \ \}$$

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$$\equiv \quad \{ \ \text{structural equality} \ \}$$

$$\langle \mathsf{at}_\rho, \pi_1 \cdot \mathsf{m}_\rho \rangle = \langle \mathsf{at}_p \cdot \pi_1, \mathsf{m}_p \cdot \pi_1 \rangle \ \wedge \ \langle \mathsf{at}_\rho, \pi_2 \cdot \mathsf{m}_\rho \rangle = \langle \mathsf{at}_q \cdot \pi_2, \mathsf{m}_q \cdot \pi_2 \rangle$$

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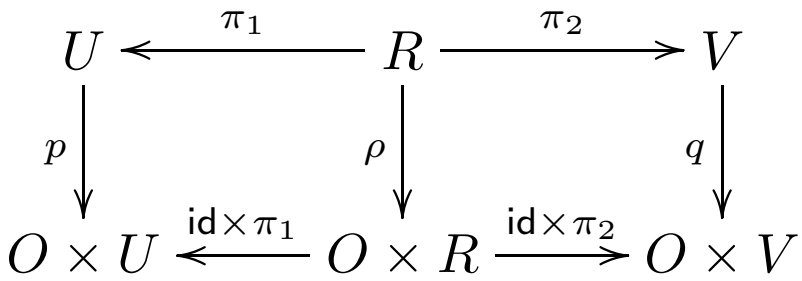
$$(\mathsf{id} \times \pi_1) \cdot \langle \mathsf{at}_\rho, \mathsf{m}_\rho \rangle = \langle \mathsf{at}_p, \mathsf{m}_p \rangle \cdot \pi_1 \ \wedge \ (\mathsf{id} \times \pi_2) \cdot \langle \mathsf{at}_\rho, \mathsf{m}_\rho \rangle = \langle \mathsf{at}_q, \mathsf{m}_q \rangle \cdot \pi_2$$

Looking for Genericity

which asserts the commutativity of



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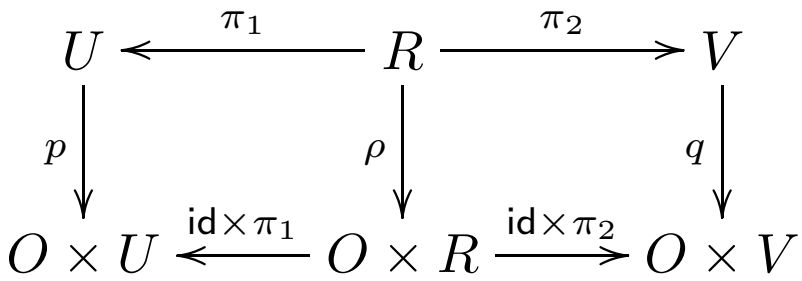
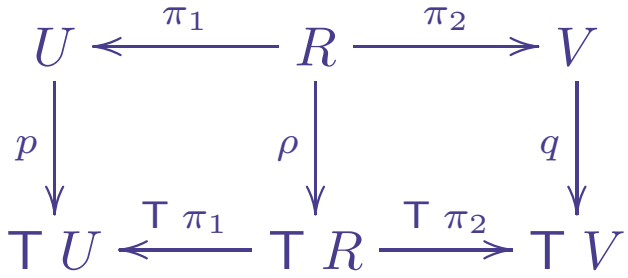


for an arbitrary functor T:



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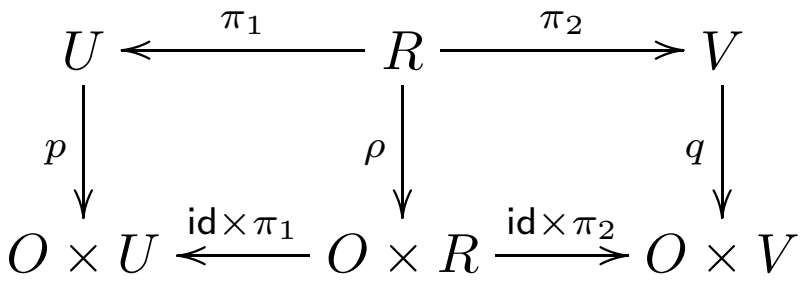

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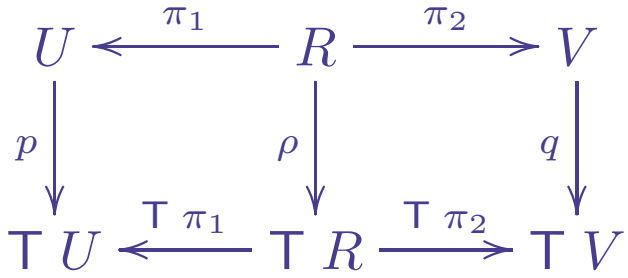

$$\mathsf{T}\ \pi_1 \cdot \rho = p \cdot \pi_1 \ \wedge\ \mathsf{T}\ \pi_2 \cdot \rho = q \cdot \pi_2$$

Example: $\mathsf{T} = \mathsf{Id}$

Any relation preserving p and q dynamics is a bisimulation.

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Because

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$\equiv$ { structural equality}

$$\langle \pi_1 \cdot \rho, \pi_2 \cdot \rho \rangle \ = \ \langle p \cdot \pi_1, q \cdot \pi_2 \rangle$$

$\equiv$ { $\times$ fusion and absorption}

$$\langle \pi_1, \pi_2 \rangle \cdot \rho \ = \ p \times q \cdot \langle \pi_1, \pi_2 \rangle$$

$\equiv$ { $\times$ reflection }

$$\rho \ = \ p \times q$$

$\equiv$ { going pointwise }

$$\langle u, v \rangle \in R \ \text{ iff } \ u \in U \ \wedge \ v \in V$$

Example: $\mathsf{T} = \mathcal{P}(O \times \mathsf{Id})$

A similar calculation leads to

$$\langle u, v\rangle \in R \;\Rightarrow$$

$$\forall_{\langle o,x\rangle \in p\, u} \cdot \exists_{\langle o,y\rangle \in q\, v} \cdot \langle x, y\rangle \in R$$

$$\wedge$$

$$\forall_{\langle o,y\rangle \in q\, v} \cdot \exists_{\langle o,x\rangle \in p\, u} \cdot \langle x, y\rangle \in R$$

which corresponds to CCS strict bisimulation

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- Modal reasoning (again parametric on system’s interface T) [Moss99, Jacobs99, Kurz01, ...]

Coinductive Programming

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- replaces the Cartesian closedness requirement, implicit in the original approaches to categorical types, by the assumption that all datatypes are strong
- i.e., the underlying functor T possess a strength [Koc72,CS92]

$$\tau_r^{\mathsf{T}} : \mathsf{T} \times - \Longrightarrow \mathsf{T}(\mathsf{Id} \times -)$$

whose effect is to distribute context along functor T

The CHARITY Language

Primitive Types

final object 1

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- functions are not values and composition, instead of application, is the fundamental primitive
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- ... either lazily or eagerly depending on the types involved being coinductive or inductive, respectively

The CHARITY Language

Coinductive Types

$$\texttt{data}\ S\texttt{->}T\texttt{(}A\texttt{)} = o_1\texttt{:}S\texttt{->}E_1(A,S)\ \texttt{|}\ \ldots\ \texttt{|}\ o_n\texttt{:}S\texttt{->}E_n(A,S)\texttt{.}$$

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Formally, $T(A)$ is the final coalgebra for a functor T_A determined by a signature of observers:

$$\langle \nu_{\mathsf{T}_A}, \langle o_1, \ldots o_n \rangle : \nu_{\mathsf{T}_A} \longrightarrow \prod_i E_i(A, \nu_{\mathsf{T}_A}) \rangle$$

parametric on A

The CHARITY Language

Examples of Coinductive Types

```
data S -> stream(X) = head: S -> X
                    | tail: S -> S.

data S -> colist(X) = delist: S -> SF(X * S).

data S -> conat = denat: S -> SF(S).

data U -> iTree A = node: U -> A
                  | rg  : U -> U
                  | lf  : U -> U.
```

The CHARITY Language

Examples of Coinductive Types

```
data S -> moor(I,O) = att : S -> O
                    | met : S -> I => S.


data S -> mealy(I,O) = a: S -> I => S * O.
```

The CHARITY Language

Inductive Types

$$\texttt{data}\ T(A) \texttt{->} S = c_1 : E_1(A, S) \texttt{->} S \mid \ldots \mid c_n : E_n(A, S) \texttt{->} S.$$

The CHARITY Language

Inductive Types

$\texttt{data}\ T(A) \texttt{->} S = c_1 \texttt{:} E_1(A,S) \texttt{->} S \ \texttt{|}\ \ldots \ \texttt{|}\ c_n \texttt{:} E_n(A,S) \texttt{->} S\texttt{.}$

$T(A)$ is the initial algebra (again parametric on A) for the functor determined by a signature of constructors:

$$\langle \mu\mathsf{T_A}, [c_1, \ldots, c_n] : \sum_i E_i(A, \mu\mathsf{T_A}) \longrightarrow \mu\mathsf{T_A} \rangle$$

The CHARITY Language

Examples of Inductive Types

```
data SF A -> C = ss: A -> C
               | ff: 1 -> C.

data nat -> C = zero: 1 -> C
              | succ: C -> C.

data list A -> C = nil : 1     -> C
                 | cons: A * C -> C.

data bTree (A, B) -> C = leaf: A           -> C
                       | node: B * (C * C) -> C.
```

The CHARITY Language

Combinators (for coinductive types)

unfold (strong anamorphism)

$$(s, c) \Rightarrow (\,|\; s \Rightarrow o_1 : p_1(s, c) \;|\; \cdots \;|\; o_n : p_n(s, c) \;|\,)$$

where

$$p_i : S \times C \longrightarrow E_i(A, S)$$

where S is the carrier of the source coalgebra and C the context type.

The CHARITY Language

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record (non-recursive unfold)

$$c \Rightarrow (\; o_1 : f_1(c) \;\cdots\; |\; o_n : f_n(c) \;)$$

The CHARITY Language

Examples

```
def gen: int  -> stream(int)
    = w => (| x => head: x
             |      tail: x
             |) w.

def stream_gen{f}: X -> stream(X)
    = x => (| x => head: x
             |      tail: f x
             |) x.
```

The CHARITY Language

Examples

```
def twist: O * O  -> stream(O)
    = w => (| (x,y) => head: x
              |      tail: (y,x)
              |) w.

def merge: stream(O) * stream(O)  -> stream(O)
    = w => (| (x,y) => head: head x
              |      tail: (y, tail x)
              |) w.
```

The CHARITY Language

```
def fibb: int * int  -> stream(int)
 = x => (| (m,n) => head: m
          |           tail: (n, add(m,n))
          |) x.

def genfac: int * int -> stream(int)
 =  (n,m)  => (| (x,y) =>
                |   head: x
                |   tail: (mul(x,y), add(y,1))
                |) (n, m).

def factorial: 1 -> stream(int)
    =  ()  =>  genfac(1,1).
```

The CHARITY Language

Combinators (for inductive types)

fold (strong catamorphism)

$$(s, c) \Rightarrow \{|\ c_1 : s_1 \Rightarrow d_1(s_1, c) |\ \cdots\ |\ c_n : s_n \Rightarrow d_n(s_n, c)\ |\} s$$

where

$$d_i : E_i(A, S) \times C \longrightarrow S$$

where C is the context and S the carrier of the target algebra: 'the *either* of all such d_i'

The CHARITY Language

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case (non-recursive fold)

$$(s, c) \Rightarrow \{\ c_1 s_1 \Rightarrow d_1(s_1, c)\ |\ \cdots\ |\ c_n s_n \Rightarrow\ \ d_n(s_n, c)\ \}\ s$$

The CHARITY Language

Examples

```
def len: list A -> nat
    = l       => {| nil : ()      => zero
                  | cons: (_, n) => succ n
                  |} l.
```

```
def reduce{oper: M * M -> M, e: 1 -> M}: list(M) -> M
 =  l  =>  {| nil: ()       => e
             | cons: (a,pr) => oper(a,pr)
             |}  l.
```

The CHARITY Language

Combinators (for both)

map (functorial extension)

$$(t, c) \Rightarrow T\{\ x_1 \Rightarrow h_1(x_1, c) \mid \cdots \mid x_m \Rightarrow h_m(x_m, c)\ \}\ t$$

for $h_i : A_i \times C \longrightarrow A'_i$