A Brief Introduction to Categories

(Lecture Notes for *Álgebra de Processos*)

Mestrado em Matemática Computacional

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1. INTRODUCTION. In the introductory material of [McL92], Colin McLarty comments on the development of category theory as follows:

> *The spread of applications led to a general theory, and what had been a tool for handling structures became more and more a means of defining them. (...) In the 1960s, Lawvere began to give purely categorical descriptions of new and old structures, and developed several styles of categorical foundations for mathematics. This lead to new applications, notably in logic and computer science.*

The notion of a *coalgebra*, which pervades this course, is one of such old-new structures, easily recognised in a variety of computational phenomena, but whose generality and expressive power became clear only under the light of category theory. This text offers an introduction to basic category theory by presenting some core concepts, notation and laws.

Our exposition of background material expresses a personal way of understanding the basic concepts and results reviewed. The reader is referred to standard textbooks, namely [Mac71] and [Bor94], for proofs and a more detailed exposition. [BW90] and [Wal91] are intended for a computer science audience, but several other introductory texts exist. One we have found particularly sharp and clear is [McL92]. On the other hand, to build up one's own intuitions, [LS97] remains a 'conceptual' pearl.

2. CATEGORIES. Roughly speaking, categories deal with *arrows* and their composition, in the same sense that sets deal with *elements*, their aggregation and membership. An *arrow* is an abstraction of the familiar notion of a function in set theory or of a homomorphism in algebra. Depicted as $f : X \longrightarrow Y$, it may be thought of as a transformation, or, simply, a connection, between two *objects* X and Y, called its *source* (or domain) and *target* (or codomain), respectively. The sources and targets of all the arrows in a category, form the class of its *objects*. If the same object is both the target of an arrow f and the source of another arrow g, f and g are said to be composable. Arrow composition is thus a partial operation and what the axioms for a category say is that arrows and arrow composition form a sort of generalized monoid. Formally,

3. Definition. A *category* C consists of

- a class $\mathrm{obj}(\mathsf{C})$ of *objects* X, Y, Z, ...
- for each pair of objects X and Y, a set of *arrows* with source X and target Y, called a *hom set* and denoted by $\mathsf{C}[X, Y]$.
- for each triple of objects X, Y and Z, an operation, called C-*composition*,

$$\cdot : \mathsf{C}[Y, Z] \times \mathsf{C}[X, Y] \longrightarrow \mathsf{C}[X, Z]$$

such that

- C-composition is *associative* and there exists, for each object X, a special arrow id_X (or simply X if no confusion with the object X itself arises), called the *identity on* X, which is both a pre- and post-unit of composition. This is neatly expressed by the commutativity of the following diagrams:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "g\cdot f"'] & Y \arrow[d, "g"] \arrow[dr, "h\cdot g"] & \\
& Z \arrow[r, "h"] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "f"'] & Y \arrow[d, "\mathsf{id}_Y"] \arrow[dr, "g"] & \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

i.e., by

$$h \cdot g \cdot f = h \cdot (g \cdot f) \tag{1}$$

$$\mathsf{id}_Y \cdot f = f \tag{2}$$

$$g \cdot \mathsf{id}_Y = g \tag{3}$$

- Arrows uniquely determine their source and target objects, *i.e.*, for all X and Y, the sets $\mathsf{C}[X, Y]$ are pairwise disjoint.

4. Examples. A prime example of a category is Set, the category of sets and set-theoretic functions. Another example is Rel in which functions are replaced by binary relations. Composition is then relational product: given arrows $r \subseteq Y \times Z$ and $s \subseteq X \times Y$, their composite is the relation $r \circ s = \{\langle x, z\rangle \subseteq X \times Z |\ \exists_{y \in Y} \ .\ \langle x, y\rangle \in s \ \wedge\ \langle y, z\rangle \in r\})$. Other typical examples of a category arise by considering algebraic structures and their homomorphisms or, even, by looking at a particular such structure as a category itself. For example, a poset forms a category by taking as arrows all the pairs of elements x, y such that $x \leq y$. Composition and identities are given, respectively, by the transitivity and reflexivity of the order relation.

There are also several standard ways of obtaining new categories from old. The simplest one consists of formally reversing all the arrows of a category C. The result, denoted by C^{op}, is referred to as the *dual category* of C. Another useful construction is the *product category*: given two categories C and D, their product category has as objects (resp. arrows) pairs $\langle C, D\rangle$ (resp. $\langle f, g\rangle$) of a C and a D-object (resp. arrow). Identities and composition are defined componentwise.

5. Universality. If there is a 'main topic' in category theory, this is certainly the study of *universal* properties. Roughly speaking, an entity ϵ is universal among a family of 'similar' entities if it is the case that every other entity in the family can be *reduced* or *traced back* to ϵ. For example, an object T is said to be *final* in a category C if, from every other object X in C, there exists a unique arrow $!_X$ to T. Therefore, there is a canonical, in the sense of *unique*, way to relate every object in C to T — *finality* is thus an universal property.

A nice thing about universal properties is the fact they always 'come in pairs': the *dual* of an universal is still an universal. Dualizing finality, we arrive at *initiality*: an object is *initial* in C if there is one and only one arrow in C from it to any other object in the category.

Universal properties, like finality or initiality, can be recognised, usually under a different terminology, in many branches of Mathematics. Moreover, they happen to play a major role in the structure of 'mathematical spaces'. Therefore, category theory provides a setting for studying abstractly such 'spaces' and their relationships.

6. FINAL AND INITIAL OBJECTS. In Set the empty set has exactly the properties of an initial object. On the other hand, if we seek for final objects, we will end up recognising that any *singleton* set will do. The usual notation for the empty set is $\emptyset$, a symbol close to 0. To stress the duality, any (actually, the) singleton will be denoted by $\mathbf{1}$. The same symbols will be used for initial and final objects in any category. The corresponding universal properties state the existence, for each object X in the category, of two unique arrows $!_X : X \longrightarrow \mathbf{1}$ and $?_X : \emptyset \longrightarrow X$ such that, for any $f : X \longrightarrow Y$:

$$!_Y \cdot f = !_X \tag{4}$$

$$f \cdot ?_X = ?_Y \tag{5}$$

7. POINTS. One way of thinking of an arrow $x : Z \longrightarrow X$ is as an 'element' of X, which is not given once and for all, but depends on Z. Following [McL92], we shall call x a *generalized element* of X and Z its *stage of definition.* This suggests the alternative notation $x \in_Z X$ for arrow $x : Z \longrightarrow X$. The composite $f \cdot x$, for $f : X \longrightarrow Y$, can thus be written as $f\ x$.

A special kind of elements of an object X consists of arrows into X whose source is the final object $\mathbf{1}$ (§6) in the category (if it exists). They are called *points* (or *global elements*) of X. In some categories every arrow $f : X \longrightarrow Y$ is fully determined by its effect on the points of X. Should this be the case, the category is said to be *well pointed.* Again Set is a good example: being well pointed is just a categorical version of the well known fact that 'a set is determined by its elements'. In Set we shall make explicit the correspondence between elements $x \in X$ and points $\underline{x} : \mathbf{1} \longrightarrow X$, by denoting function application by juxtaposition, *i.e.*, $fx = f \cdot \underline{x}$.

8. ISOMORPHISM. What is 'inside' $\mathbf{1}$? More generally, in what sense an universal entity is, in fact, *the* universal? In category theory, elements and extensionality have been abstracted away and therefore the internal structure of C objects is not available when reasoning at the level of C. Objects that cannot be distinguished in the language of category theory are called *isomorphic.* Think, for a moment in sets like $\{0\}$, $\{20\}$, $\{\mathtt{abc}\}$ or $\{*\}$. From a categorical point of view they have to be characterised in terms of their behaviour under ingoing or outgoing arrows. And it turns out that all that can be said of them, from this point of view, is that they fulfil the properties of a final object. As it is not possible to distinguish them further, the symbol $\mathbf{1}$ has been taken to denote their isomorphism class. In other words, we may say that each such set is the final element in Set *up to isomorphism.* In fact, all objects defined by universal constructions are unique up to isomorphism.

9. DEFINITION. The very notion of isomorphism is stated in the language of arrows. An arrow $f : X \longrightarrow Y$ is an *isomorphism* if there is another arrow $g : Y \longrightarrow X$ such that

$$f \cdot g = \mathsf{id}_Y \tag{6}$$
$$g \cdot f = \mathsf{id}_X \tag{7}$$

If such a g exists it is said the *inverse* of f and written f°. A right (resp. left) inverse to f is also called a *section* (resp. *retraction*). If there is an isomorphism between two objects X and Y, they are said to be *isomorphic* and we write $X \cong Y$.

10. CANCELLATION. A weaker requirement an arrow may satisfy is *cancellation.* Like existence of inverse, it has a right and a left version. We say $f : X \longrightarrow Y$ is *cancellable on the right*, an *epimorphism*, or, simply, an *epi*, if, for every $z_1, z_2 : Y \longrightarrow Z$,

$$z_1 \cdot f = z_2 \cdot f \;\;\Rightarrow\;\; z_1 = z_2 \tag{8}$$

Dually, it is said to be *cancellable on the left*, a *monomorphism*, or, simply, a *mono*, if, for every $x_1, x_2 : Z \longrightarrow X$,

$$f \cdot x_1 = f \cdot x_2 \;\;\Rightarrow\;\; x_1 = x_2 \tag{9}$$

Cancellation propagates through composition, in the sense that, in any category, any composition of monos (resp. epis) is still a mono (resp. an epi).

11. MONOMORPHISMS. If we think of x_1 and x_2 as generalized elements of X (§7), the definition of a *mono* above looks familiar: for all, $x_1, x_2 \in_Z X$,

$$f x_1 = f x_2 \;\;\Rightarrow\;\; x_1 = x_2 \tag{10}$$

resembles the definition of an injective function. In fact, a *mono* is an injection over generalized elements and happens to be an injection in Set and in most categories of sets with structure.

12. EPIMORPHISMS. On the other hand, an *epi* does not convey entirely the notion of a surjection. In fact, what the definition of an epi $f : X \longrightarrow Y$ says is that f covers a 'sufficiently large' part of Y, in the sense that, to be distinguished, arrows from Y must disagree somewhere in the part covered by f. The stronger property of being right invertible (6) is required to capture surjectivity. In the language of generalized elements, the latter reads,

$$\text{for each } Z \text{ and } y \in_Z Y, \text{ there exists an } x \in_Z X \text{ such that } f\ x = y$$

One can easily show that a right invertible arrow is also an epi, said a *split epi.* In Set, but not in many other cases, all epis are split.

13. FUNCTORS. Once a structure has been introduced, the natural next step one gets used to from Universal Algebra, is to look for an appropriate definition of a morphism that preserves such a structure. A *functor* is exactly such a morphism for categories (*i.e.*, a homomorphism of categories). Therefore, it preserves the typing of arrows and identities and distributes over composition. Formally,

14. Definition. A *functor* $\mathsf{T} : \mathsf{C} \longrightarrow \mathsf{D}$ from a category C to a category D is a mapping assigning to each $X \in \mathrm{obj}(\mathsf{C})$ an object $\mathsf{T}\ X$ in $\mathrm{obj}(\mathsf{D})$ and, similarly, to each $f \in \mathsf{C}[X, Y]$ an arrow $\mathsf{T}\ f \in \mathsf{D}[\mathsf{T}\ X, \mathsf{T}\ Y]$ such that

$$\mathsf{T}\ \mathsf{id}_X = \mathsf{id}_{\mathsf{T}\ X} \tag{11}$$

$$\mathsf{T}\ (f \cdot g) = \mathsf{T}\ f \cdot \mathsf{T}\ g \tag{12}$$

15. Cat. As expected, functors with compatible typing can be composed. Functor composition will be represented in the sequel by $\circ$ or, more often, by juxtaposition. Moreover, for each category C there is a an identity functor denoted by C or Id_{C} (or simply Id) which is the identity on both objects and arrows. Therefore, categories and functors form themselves a category Cat.

16. Special Functors. A functor from a category C to itself is said to be an *endofunctor*. On the other hand, a functor whose source is a product category is called a *bifunctor* and is often represented by an infix operator. Given a bifunctor $\mathsf{T} : \mathsf{C} \times \mathsf{D} \longrightarrow \mathsf{E}$ and an object C of C, a *C-section* of T is the functor $\mathsf{T}_C : \mathsf{D} \longrightarrow \mathsf{E}$ obtained from T by fixing its first argument.

Associated with each object $X \in \mathrm{obj}(\mathsf{C})$, there is also the *constant* functor on X which maps every object to X and every morphism to the identity on X. This functor will be simply written X (or $\underline{X}$, if the context is ambiguous).

17. Natural Transformations. Functors can be seen not only as arrows in Cat, but also as objects of other categories, provided a suitable notion of morphism is defined. This is exactly what a *natural transformation* is. Formally,

18. Definition. Given two functors $\mathsf{T}, \mathsf{S} : \mathsf{C} \longrightarrow \mathsf{D}$ a *natural transformation* $\sigma : \mathsf{T} \Longrightarrow \mathsf{S}$ is a family of D-arrows, indexed by the objects of C, such that, for any C-arrow $f : X \longrightarrow Y$ the following diagram commutes:

$$\begin{tikzcd}
X \arrow[d, "f"'] & \mathsf{T}\ X \arrow[r, "\sigma_X"] \arrow[d, "\mathsf{T}\ f"'] & \mathsf{S}\ X \arrow[d, "\mathsf{S}\ f"] \\
Y & \mathsf{T}\ Y \arrow[r, "\sigma_Y"'] & \mathsf{S}\ Y
\end{tikzcd}$$

i.e.,

$$\sigma_Y \cdot \mathsf{T}\ f = \mathsf{S}\ f \cdot \sigma_X \tag{13}$$

Each σ_X is referred to as the *component* of σ at the object X.

19. Vertical Composition. Suppose T, S and R are functors from C to D and that there are natural transformations $\sigma : \mathsf{T} \Longrightarrow \mathsf{S}$ and $\sigma' : \mathsf{S} \Longrightarrow \mathsf{R}$. Then, σ and σ' can be composed originating $\sigma' \cdot \sigma : \mathsf{T} \Longrightarrow \mathsf{R}$, by defining

$$(\sigma' \cdot \sigma)_X = \sigma'_X \cdot \sigma_X \tag{14}$$

This is known as the *vertical* composition of natural transformations. Thus, we may form the *functor category*, D^{C}, of functors from C to D and natural transformations. Notice that, for each functor T in D^{C}, the family of identity arrows $\mathsf{id}_{\mathsf{T}\,X}$ in D gives rise to a trivial natural transformation denoted by 1_{T}, which acts as an identity in D^{C}.

20. Horizontal Composition. There is also a notion of composition for natural transformations between pairs of composable functors. It will be denoted by by **;** and used in diagrammatic order. Suppose T and S are functors from C to D and T' and S' are functors from D to E. If there exist natural transformations $\sigma : \mathsf{T} \Longrightarrow \mathsf{S}$ and $\sigma' : \mathsf{T}' \Longrightarrow \mathsf{S}'$, their *horizontal* composite is $\sigma \,;\, \sigma' : \mathsf{T}'\,\mathsf{T} \Longrightarrow \mathsf{S}'\,\mathsf{S}$ whose component at X is given by

$$(\sigma\,;\,\sigma')_X \;= \mathsf{S}'\,\sigma_X \cdot \sigma'_{\mathsf{T}\,X} \;=\; \sigma'_{\mathsf{S}\,X} \cdot \mathsf{T}'\,\sigma_X \qquad (15)$$

as, by definition of σ and σ', the following diagram commutes:

$$\begin{tikzcd}
\mathsf{T}'\,\mathsf{T}\,X \arrow[r, "\sigma'_{\mathsf{T}\,X}"] \arrow[d, "\mathsf{T}'\,\sigma_X"'] \arrow[dr, "(\sigma;\sigma')_X"] & \mathsf{S}'\,\mathsf{T}\,X \arrow[d, "S'\,\sigma_X"] \\
\mathsf{T}'\,\mathsf{S}\,X \arrow[r, "\sigma'_{\mathsf{S}\,X}"'] & \mathsf{S}'\,\mathsf{S}\,X
\end{tikzcd}$$

Particular cases of this situation occur when σ or σ' are the identity 1_{R} on a functor R. We may, then, pre- or post-compose σ with 1_{R}, yielding

$$\mathsf{R}\sigma \overset{\text{abv}}{=} \sigma\,;\,1_{\mathsf{R}} : \mathsf{R}\,\mathsf{T} \longrightarrow \mathsf{R}\,\mathsf{S} \qquad \text{with} \quad (\mathsf{R}\sigma)_X = \mathsf{R}\,\sigma_X \qquad (16)$$

$$\sigma\mathsf{R} \overset{\text{abv}}{=} 1_{\mathsf{R}}\,;\,\sigma : \mathsf{T}\,\mathsf{R} \longrightarrow \mathsf{S}\,\mathsf{R} \qquad \text{with} \quad (\sigma\mathsf{R})_X = \sigma_{\mathsf{R}\,X} \qquad (17)$$

where $\mathsf{T}, \mathsf{S} : \mathsf{C} \longrightarrow \mathsf{D}$, $\sigma : \mathsf{T} \Longrightarrow \mathsf{S}$ and R is a functor from D to E, in the first case, and from B to C, in the second. Vertical and horizontal composition of natural transformations interact via the *interchange law.*

21. From a programming point of view, it is remarkable that the basic properties captured in the categorical framework — such as *universality*, *functoriality* and *naturality* — can be phrased in a 'calculational' style. This means that such properties can be formulated as (usually equational) laws and used to manipulate and reason about objects and arrows of the underlying category. Such a 'calculational' style matches nicely a main concern in computer science — the seek for program calculi able to promote programming to a modern engineering discipline.

In the next paragraphs we will recall the *product* and *coproduct* constructions and some associated laws that turn out to be most useful in calculation. Product and coproduct are the categorical generalisation of Cartesian product and disjoint union in Set. In a sense, they capture the duality between *co-occurrence* and *choice*, which may explain their major role in modelling computational systems. From §24 on, universal constructions, like product and coproduct, or final and initial object, will be revisited in this broader, more comprehensive setting.

22. Product. The *product* of two objects X and Y in a category C is an object $X \times Y$ defined as the source of two arrows $\pi_1 : X \times Y \longrightarrow X$ and $\pi_2 : X \times Y \longrightarrow Y$, called the *projections*, which satisfy the following universal property: for any other $Z \in \mathrm{obj}(\mathsf{C})$ and arrows $f : Z \longrightarrow X$ and $g : Z \longrightarrow Y$, there is a unique arrow $\langle f, g\rangle : Z \longrightarrow X \times Y$, usually called the *split* of f and g, that makes the following diagram to commute:

$$\begin{tikzcd}
& Z \arrow[dl, "f"'] \arrow[d, "\langle f,g\rangle"] \arrow[dr, "g"] & \\
X & X \times Y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y
\end{tikzcd}$$

This universal property can be written as

$$k = \langle f, g\rangle \;\Leftrightarrow\; \pi_1 \cdot k = f \,\wedge\, \pi_2 \cdot k = g \tag{18}$$

where $\Rightarrow$ means *existence* and $\Leftarrow$ means *uniqueness*. Form (18) the following laws are easily derived:

$$\pi_1 \cdot \langle f, g\rangle = f \;,\; \pi_2 \cdot \langle f, g\rangle = g \tag{19}$$

$$\langle \pi_1, \pi_2\rangle = \mathsf{id}_{X \times Y} \tag{20}$$

$$\langle g, h\rangle \cdot f = \langle g \cdot f, h \cdot f\rangle \tag{21}$$

which exemplify, for the product construction, what is sometimes called a *cancellation*, *reflection* and *fusion* result, respectively. Structural equality is also derivable from (18):

$$\langle f, g\rangle = \langle k, h\rangle \;\equiv\; f = k \,\wedge\, g = h \tag{22}$$

23. Coproduct. The *sum*, or *coproduct*, of X and Y in a category C is the dual construction — actually it simply is the product in C^{op}. That is to say, an object $X + Y$ defined as the target of two arrows $\iota_1 : X \longrightarrow X+Y$ and $\iota_2 : Y \longrightarrow X+Y$, called the *injections*, which satisfy the following universal property: for any other $Z \in \mathrm{obj}(\mathsf{C})$ and arrows $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$, there is a unique arrow $[f, g] : X + Y \longrightarrow Z$, usually called the *either* (or *case*) of f and g, that makes the following diagram to commute:

$$\begin{tikzcd}
X \arrow[r, "\iota_1"] \arrow[dr, "f"'] & X + Y \arrow[d, "{[f,g]}"] & Y \arrow[l, "\iota_2"'] \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

Again this universal property can be written as

$$k = [f, g] \;\Leftrightarrow\; k \cdot \iota_1 = f \,\wedge\, k \cdot \iota_2 = g \tag{23}$$

from which one infers correspondent *cancellation*, *reflection* and *fusion* results:

$$[f,g]\cdot\iota_1 = f \ , \ [f,g]\cdot\iota_2 = g \tag{24}$$

$$[\iota_1,\iota_2] \ = \ \mathsf{id}_{X+Y} \tag{25}$$

$$f\cdot[g,h] \ = \ [f\cdot g, f\cdot h] \tag{26}$$

Products and sums interact through the following *exchange* law

$$[\langle f,g\rangle,\langle f',g'\rangle] \ = \ \langle[f,f'],[g,g']\rangle \tag{27}$$

provable by either $\times$ (18) or $+$ (23) universality.

24. UNIVERSALITY REVISITED. If categories can be thought of as particular mathematical spaces and functors as structure-preserving translations between them, an *adjunction* between, say, two functors $\mathsf{T} : \mathsf{C} \longrightarrow \mathsf{D}$ and $\mathsf{S} : \mathsf{D} \longrightarrow \mathsf{C}$, can be regarded as a source of *universals* in C and D. In fact, products and coproducts, final and initial objects and, in general, any universal construction arise in such a context. The conceptual relevance of the notion of an adjunction, which pervades category theory and, in a sense, Mathematics as a whole, justifies a somewhat more detailed introduction here. We will then revisit products and sums and introduce function spaces from the categorical point of view.

We have said in §5 that, by an entity being universal among a collection of similar ones, it is understood that there exists a unique way in which every other entity in the collection can be reduced to (or factored through) it. It turns out that a very general way to express such collections is as families of arrows, said *from* T *to* Z,

$$\{f : \mathsf{T}\ X \longrightarrow Z \mid X \in \mathrm{obj}(\mathsf{C})\}$$

for T a functor from C to D and Z an object of D.

An arrow ϵ_Z in such a family is *universal* if there is, for every $f : \mathsf{T}\ X \longrightarrow Z$, a unique arrow $f^\bullet : X \longrightarrow \mathsf{S}\ Z$ in C, such that f factors through ϵ_Z as expressed by the commutativity of the following diagram:

$$\begin{tikzcd}
X \arrow[d, "f^\bullet"'] & & \mathsf{T}\ X \arrow[d, "\mathsf{T}\ f^\bullet"'] \arrow[dr, "f"] & \\
\mathsf{S}\ Z & & \mathsf{T}\ \mathsf{S}\ Z \arrow[r, "\epsilon_Z"'] & Z
\end{tikzcd}$$

i.e., $f = \epsilon_Z \cdot \mathsf{T}\ f^\bullet$.

For the moment, think of $\mathsf{S}\ Z$ just as an object of C, related to Z in a way that will become clear soon. This object, as well as ϵ_Z, is of course unique (up to isomorphism). Finally, notice how the intuitive idea of *reducing* is formally captured in the notion of *factorisation*.

25. PRODUCT REVISITED. Let us instantiate this definition with a familiar example. Take T as $\triangle$: $\mathsf{C} \longrightarrow \mathsf{C}\times\mathsf{C}$, the *diagonal* functor from C to its product category. Let $Z = \langle Z_1, Z_2\rangle$ be an object of $\mathsf{C}\times\mathsf{C}$. If it exists, an universal arrow from $\triangle$ to Z will be a pair $\epsilon_Z = \langle p_1, p_2\rangle$ of arrows such that, for each $\mathsf{C}\times\mathsf{C}$ arrow

$$f = \langle f_1, f_2\rangle : \triangle\ X \longrightarrow Z$$

there exists a unique $f^\bullet$ such that

$$\langle f_1, f_2\rangle = \langle p_1, p_2\rangle \cdot \triangle\ f^\bullet$$

Diagrammatically,

$$\begin{tikzcd}
X \arrow[d, "f^\bullet"'] & & \triangle\ X \arrow[d, "\triangle\ f^\bullet"'] \arrow[dr, "f=\langle f_1,f_2\rangle"] & \\
\mathsf{S}\ Z & & \triangle\ \mathsf{S}\ Z \arrow[r, "\langle p_1,p_2\rangle"'] & Z
\end{tikzcd}$$

As composition on $\mathsf{C} \times \mathsf{C}$ is defined componentwise, this can be written as

$$f_1 = p_1 \cdot f^\bullet \quad \text{and} \quad f_2 = p_2 \cdot f^\bullet$$

It is then straightforward to recognise $\mathsf{S}\ Z$ as the product $Z_1 \times Z_2$, p_1 and p_2 as the associated projections π_1 and π_2, and $f^\bullet$ as the *split* of f_1 and f_2. We, therefore, conclude that the pair $\langle \pi_1, \pi_2\rangle$ is the universal arrow in the family $\{f : \triangle\ X \longrightarrow \langle Z_1, Z_2\rangle \mid X \in \mathrm{obj}(\mathsf{C})\}$ and, for any f the *split* $\langle f_1, f_2\rangle$ of its components is the induced unique arrow.

26. PRODUCT AS A FUNCTOR. If the construction mentioned in the previous paragraph can be repeated for every object in $\mathsf{C} \times \mathsf{C}$, we get along all the binary products on C. Moreover, S emerges as a correspondence between pairs of C-objects and their product, which can be made *functorial* as follows. Let $\langle h_1, h_2\rangle : \langle A_1, A_2\rangle \longrightarrow \langle B_1, B_2\rangle$ be a morphism in $\mathsf{C} \times \mathsf{C}$. Then, define,

$$h_1 \times h_2 \;=\; (\langle h_1, h_2\rangle \cdot \epsilon_{\langle A_1, A_2\rangle})^\bullet$$

and simplify

$$\begin{aligned}
& (\langle h_1, h_2\rangle \cdot \epsilon_{\langle A_1, A_2\rangle})^\bullet \\
= \quad & \quad \{\ \epsilon_{\langle A_1, A_2\rangle} \text{ definition }\} \\
& (\langle h_1, h_2\rangle \cdot \langle \pi_1, \pi_2\rangle)^\bullet \\
= \quad & \quad \{\ \text{composition in } \mathsf{C} \times \mathsf{C}\ \} \\
& (\langle h_1 \cdot \pi_1, h_2 \cdot \pi_2\rangle)^\bullet \\
= \quad & \quad \{\ f^\bullet \text{ definition }\} \\
& \langle h_1 \cdot \pi_1, h_2 \cdot \pi_2\rangle
\end{aligned}$$

Back into the general case, note that in §24 we have been rather vague about construction $\mathsf{S}\ Z$. Provided universal arrows ϵ can be defined for each object of C, the answer is now obvious: S is a functor from D to C, whose action $\mathsf{S}\ h$, on a D-morphism $h : A \longrightarrow B$, is defined by the following diagram,

$$\begin{tikzcd}
\mathsf{S}\ A \arrow[d, "\mathsf{S}\ h=(h\cdot\epsilon_A)^\bullet"'] & & \mathsf{T}\mathsf{S}\ A \arrow[d, "\mathsf{T}\ (h\cdot\epsilon_A)^\bullet"'] \arrow[dr, "h\cdot\epsilon_A"] & \\
\mathsf{S}\ B & & \mathsf{T}\ \mathsf{S}\ B \arrow[r, "\epsilon_B"'] & B
\end{tikzcd}$$

But let us come back again to the example at hands. Once product has been found functorial, the kit of laws introduced in §22 is automatically extended with the functorial laws (11) and (12), as well as a derived result showing a product of two arrows being 'absorbed' by a *split* of other two:

$$(i \times j) \cdot \langle g, h \rangle \;=\; \langle i \cdot g, j \cdot h \rangle \tag{28}$$

27. Universal Becomes Natural. Another consequence of the existence of universal arrows ϵ_Z from T to Z, for *each* object $Z \in \mathrm{obj}(\mathsf{D})$, is the emergence of a natural transformation

$$\epsilon : \mathsf{TS} \Longrightarrow \mathsf{Id}$$

that is, for the product case,

$$\epsilon : \vartriangle \times \Longrightarrow \mathsf{Id}_{\mathsf{C} \times \mathsf{C}}$$

i.e.,

$$\pi_1 \cdot (f \times g) = f \cdot \pi_1 \quad \text{and} \quad \pi_g \cdot (f \times g) = g \cdot \pi_2 \tag{29}$$

Another natural transformation which happens to play a complementary role with respect to ϵ_Z, is defined by considering the family of C-arrows corresponding to identities on objects $\mathsf{T}\ X$:

$$\eta : \mathsf{Id} \Longrightarrow \mathsf{ST}$$

For products, we have

$$\eta : \mathsf{Id}_{\mathsf{C}} \Longrightarrow \times \vartriangle$$

whose component, for each object X of C, is

$$\eta_X \;=\; (\mathsf{id}_{\vartriangle X})^{\bullet} : X \longrightarrow \times \vartriangle X$$

28. The Dual Picture. Each η_X can also be described as an universal arrow in the family of arrows

$$\{g : X \longrightarrow \mathsf{S}\ Z |\ Z \in \mathrm{obj}(\mathsf{D})\}$$

for S a functor from D to C and X an object of C.

Being universal in such a family of arrows (said *from X to* S) means, again, the existence of a unique factorisation. I.e., for all $g : X \longrightarrow \mathsf{S}\ Z$, there is a unique arrow $g_\bullet : \mathsf{T}\ Y \longrightarrow X$ such that $g = \mathsf{S}\ g_\bullet \cdot \eta_X$. In a diagram,

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "g"'] & \mathsf{ST}\ X \arrow[d, "\mathsf{S}\ g_\bullet"] & & \mathsf{T}\ X \arrow[d, "g_\bullet"] \\
& \mathsf{S}\ Z & & Z
\end{tikzcd}$$

$\mathsf{T}\ X$ can, as we have first done with $\mathsf{S}\ Z$ in §24, be thought of as a mere object of D depending on X. However, as η is defined for all objects of C, T can be made into a functor from C to D. In particular, for any $h : A \longrightarrow B$, $\mathsf{T}\ h$ is defined by the following diagram:

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "\eta_B \cdot h"'] & \mathsf{ST}\ A \arrow[d, "\mathsf{S}\ (\eta_B \cdot h)_\bullet"] & & \mathsf{T}\ A \arrow[d, "\mathsf{T}\ h = (\eta_B \cdot h)_\bullet"] \\
& \mathsf{ST}\ B & & \mathsf{T}\ B
\end{tikzcd}$$

Finally, note that, if, in the beginning of this paragraph, η has been defined in terms of the unique factorisations of identities under ϵ, the converse is also true. In fact, ϵ can be defined as a natural transformation from TS to Id_D whose components are given by $\epsilon_Z = (\mathsf{id}_{\mathsf{S}\ Z})_\bullet$. As noticed above, we end up with two 'twin' (inter-definable) natural transformations ϵ and η.

29. Example. As an example, take S as the diagonal functor $\triangle$ and seek for universal arrows from $X = \langle X_1, X_2 \rangle$ to $\triangle$. The definition reads as expected: there is an arrow $\eta_Y = \langle q_1, q_2 \rangle$ such that, for each arrow $g = \langle g_1, g_2 \rangle : X \longrightarrow_{\triangle} Z$ in $\mathsf{C} \times \mathsf{C}$, there is a unique $g_\bullet : \mathsf{T}\ X \longrightarrow Z$ such that $g =_{\triangle} g_\bullet \cdot \langle q_1, q_2 \rangle$, *i.e.*,

$$g_1 = g_\bullet \cdot q_1 \quad \text{and} \quad g_2 = g_\bullet \cdot q_2$$

Clearly, $\mathsf{T}\ X$ is the *coproduct* $X_1 + X_2$ and η_X is the pair of injections $\langle \iota_1 : X_1 \longrightarrow X_1 + X_2, \iota_2 : X_2 \longrightarrow X_1 + X_2 \rangle$. Furthermore, $\langle h_1, h_2 \rangle_\bullet$ is, by uniqueness, the *either* of h_1 and h_2. As such a construction is possible for every object in $\mathsf{C} \times \mathsf{C}$, the coproduct construction becomes a functor. Given an arrow $\langle h_1, h_2 \rangle : \langle A_1, A_2 \rangle \longrightarrow \langle B_1, B_2 \rangle$ define

$$h_1 + h_2 \;=\; (\eta_{\langle B_1, B_2 \rangle} \cdot \langle h_1, h_2 \rangle)_\bullet = \;[\iota_1 \cdot h_1, \iota_2 \cdot h_2]$$

as expected, which paves the way for the dual 'absorption' law for sums:

$$[g, h] \cdot (i + j) \;= [g \cdot i, h \cdot j] \tag{30}$$

30. Adjunctions. As we have just seen, the existence of an universal arrow from a functor $\mathsf{T} : \mathsf{C} \longrightarrow \mathsf{D}$ to every D-object Z defines uniquely (up to isomorphism, of course!) a new functor $\mathsf{S} : \mathsf{D} \longrightarrow \mathsf{C}$. This is called the *right adjoint* of T. Similarly, if there exists an universal arrow from every C-object X to functor S, a functor T is uniquely defined. T is then called the *left adjoint* of S. This kind of relation between S and T is known as an *adjunction.* Adjoint functors are written $\mathsf{T} \dashv \mathsf{S}$.

As expected, there are two dual ways of defining an adjunction. In fact, the underlying symmetry in this notion can be made explicit by observing that η is obtained by reducing identities under T to ϵ and, similarly, ϵ results from the reduction of identities under S to η. Formally,

31. Definition. A functor $\mathsf{T} : \mathsf{C} \longrightarrow \mathsf{D}$ is *left adjoint* to another functor $\mathsf{S} : \mathsf{D} \longrightarrow \mathsf{C}$, written

$$\mathsf{T} \dashv \mathsf{S}$$

if there is a natural transformation $\epsilon : \mathsf{TS} \Longrightarrow \mathsf{Id_D}$ such that, for all X in C and Z, $f : \mathsf{T}\ X \longrightarrow Z$ in D, there exists a unique arrow $g : X \longrightarrow \mathsf{S}\ Z$ such that $f = \epsilon_Z \cdot \mathsf{T}\ g$. Usually, g is written as $f^\bullet$ in order to emphasise its uniqueness upon f.

Alternatively, if there is a natural transformation $\eta : \mathsf{Id_C} \Longrightarrow \mathsf{ST}$ such that, for all X and $g : X \longrightarrow \mathsf{S}\ Z$ in C and Z in D, there exists a unique arrow $f : \mathsf{T}\ X \longrightarrow Z$ such that $g = \mathsf{S}\ f \cdot \eta_X$. Dually, f is written as $g_\bullet$.

In both cases, these (equivalent) definitions guarantee the existence of 'enough' universals. In consequence, an adjunction gives rise to a (natural) bijective correspondence between arrows $f : \mathsf{T}\ X \longrightarrow Z$, in D, and $g : X \longrightarrow \mathsf{S}\ Z$, in C, captured by the following equivalence

$$g = \mathsf{S}\ f \cdot \eta \ \Leftrightarrow \ f = \epsilon \cdot \mathsf{T}\ g \tag{31}$$

Yet another popular definition of $\mathsf{T} \dashv \mathsf{S}$ is formulated simply in terms of the following conditions (known as the 'triangle equalities') on η and ϵ:

$$\mathsf{S}\ \epsilon \cdot \eta\mathsf{T} = 1_\mathsf{S} \tag{32}$$

$$\epsilon\mathsf{T} \cdot \mathsf{T}\ \eta = 1_\mathsf{T} \tag{33}$$

In any case, η (resp. ϵ) is called the *unit* (resp. *counit*) of the adjunction.

32. LIMITS AND COLIMITS. In §§25, 29 we have discussed, in some detail, how products and coproducts arise as, respectively, left and right adjoints to the diagonal functor. In fact, adjunctions $+ \dashv \vartriangle$ and $\vartriangle \dashv \times$ are particular cases of two fundamental families of adjunctions: the ones that give rise to limits and colimits in general. A basic observation is the isomorphism between $\mathsf{D} \times \mathsf{D}$ and the functor category $\mathsf{D}^{\mathbf{2}}$, where $\mathbf{2}$ is the category with two objects and no arrows other than the associated identities. We may now generalise the notion of a diagonal functor and look for its right and left adjoints. First the generalized diagonal functor $\vartriangle_\mathsf{H}$: $\mathsf{C} \longrightarrow \mathsf{C}^\mathsf{H}$ is defined, for H is any small category. Clearly $\vartriangle_\mathbf{2}$ is the usual $\vartriangle$.

Next, take $\vartriangle_\mathbf{1}$: $\mathsf{C} \longrightarrow \mathsf{C}^\mathbf{1}$, which maps every object in C into the unique functor from $\mathbf{1}$ to C. Its right adjoint, if it exists at all, maps this unique object of $\mathsf{C}^\mathbf{1}$ into the final object of C and the corresponding η_X, for all $X \in \mathrm{obj}(\mathsf{C})$, coincides with $!_X$. Dually, a left adjoint would give the initial object and identify ϵ_X with $?_X$.

We can think of functors from H to C as H-shaped *diagrams* in C. In general, right (resp. left) adjoints to $\vartriangle_\mathsf{H}$ give the limit (resp. colimit) of such diagrams. For example, taking H as the category with three objects depicted as follows

$$\bullet \longrightarrow \bullet \longleftarrow \bullet$$

the right adjoint to $\vartriangle_\mathsf{H}$ defines *pullbacks*. Similarly, *pushouts* arise by reversing the arrows in H above and taking the left adjoint to the same 'diagonal' functor.

A basic result on adjunctions states that, in an adjunction situation $\mathsf{T} \dashv \mathsf{S}$, the left adjoint, T, preserves all colimits while, dually, the right adjoint S does the same for limits.

33. EXPONENTIALS. The categorical version of the usual notion of a function space in Set also arises, as one could expect, from an adjunction situation.

Let C be an object of C and suppose that $\mathsf{Id} \times C$, the C-section of the product bifunctor, has a right adjoint which we shall denote by Id^C. This means that for all $f : X \times C \longrightarrow Y$, there exists a unique $f^\bullet : X \longrightarrow Y^C$ such that $f = \epsilon_Y \cdot (f^\bullet \times C)$, both the object Y^C and the universal ϵ_Y being uniquely determined up to isomorphism. Diagrammatically,

$$\begin{tikzcd}
X \arrow[d, "f^\bullet"'] & & X \times C \arrow[d, "f^\bullet \times \mathsf{id}_C"'] \arrow[dr, "f"] & \\
Y^C & & Y^C \times C \arrow[r, "\epsilon_Y"'] & Y
\end{tikzcd}$$

Following the general construction of §26, Id^C extends to a functor by defining, for $h : A \longrightarrow B$,

$$h^C : A^C \longrightarrow B^C \quad = \quad (h \cdot \epsilon_A)^\bullet \tag{34}$$

Are we done? In fact, note that Y^C has exactly the characteristic properties of the set of functions from C to Y in Set. Bijection $f \leftrightsquigarrow f^\bullet$ corresponds, in this particular context, to *currying*: the well-known isomorphism between (binary) functions from $X \times C$ to Y and (unary) functions from X to the set of functions from C to Y. Being so popular, this terminology is also adopted in an arbitrary category: $f^\bullet$ is called the *curry* of f and is written $\overline{f}$.

The family $\epsilon_X : X^C \times C \longrightarrow X$ is, of course, the counit of adjunction

$$\mathsf{Id} \times C \dashv \mathsf{Id}^C$$

On the other hand, its unit has $\eta_X : X \longrightarrow (X \times C)^C$ as components. In Set, ϵ corresponds to function evaluation and η to a function constructor:

$$\begin{aligned}
\epsilon\ \langle f, c \rangle &= f\ c \\
\eta\ x &= \lambda c.\ \langle x, c \rangle
\end{aligned}$$

Counit ϵ is more commonly named ev, after *evaluation.* We shall refer to η as sp, after *stamping*, and, again, such designations will carry over to general case.

34. Exponential Laws. Adjunction $\mathsf{Id} \times C \dashv \mathsf{Id}^C$ entails an universal characterisation of exponentials:

$$k = \overline{f} \quad \Leftrightarrow \quad f = \mathsf{ev} \cdot (k \times \mathsf{id}) \tag{35}$$

from which the following laws are derived

$$\begin{aligned}
f &= \mathsf{ev} \cdot (\overline{f} \times \mathsf{id}) && (36)\\
\overline{\mathsf{ev}} &= \mathsf{id}_{X^C} && (37)\\
\mathsf{sp} &= \overline{\mathsf{id}_{X \times C}} && (38)\\
\overline{g} \cdot f &= \overline{g \cdot (f \times \mathsf{id})} && (39)
\end{aligned}$$

In an arbitrary category with exponentials C, A^C represents, as a C-object, the arrows from C to A. Consequently, the action of Id^C on each arrow $f : A \longrightarrow B$ should *internalise* composition with f. In Set it is easy to verify that this is indeed the case. For $g : C \longrightarrow A$ and $c \in C$, a simple calculation yields,

$$\begin{array}{ll}
 & (f^C\ g)\ c \\
= & \qquad \{\ \mathsf{Id}^C \text{ on arrows (34)}\ \} \\
 & (\overline{(f \cdot \mathsf{ev})}\ g)\ c \\
= & \qquad \{\ \text{uncurrying}\ \} \\
 & f \cdot \mathsf{ev}\ \langle g, c\rangle \\
= & \qquad \{\ \text{function composition}\ \} \\
 & f\ (\mathsf{ev}\ \langle g, c\rangle) \\
= & \qquad \{\ \mathsf{ev} \text{ definition}\} \\
 & f\ (g\ c) \\
= & \qquad \{\ \text{function composition}\ \} \\
 & (f \cdot g)\ c
\end{array}$$

In an arbitrary category, however, we cannot talk about ‘applying’ a morphism to an ‘element’ of an object. We have, then, to state and prove the intended result in the language of generalized elements (§7). A generalized element of an exponential A^C is an arrow $\overline{g} : T \longrightarrow A^C$, which corresponds uniquely, under the adjunction, to $g : T \times C \longrightarrow A$. Keeping in mind that, in the generalized elements notation, $f^C\ \overline{g}$ corresponds to $f^C \cdot \overline{g}$, the ‘internalisation’ result takes the form of an ‘absorption’ property for exponentials:

$$\overline{f \cdot g} \ = f^C \cdot \overline{g} \tag{40}$$

Proof. Consider the following diagram

$$\begin{tikzcd}
T \times C \arrow[r, "\overline{g}\times C"] \arrow[dr, "g"'] & A^C \times C \arrow[r, "f^C\times C"] \arrow[d, "\mathsf{ev}_A"] & B^C \times C \arrow[d, "\mathsf{ev}_B"] \\
 & A \arrow[r, "f"'] & B
\end{tikzcd}$$

and note that the left triangle commutes by definition of $\overline{g}$ and the square because ev is natural. Therefore,

$$\begin{array}{ll}
 & f \cdot g \ = \ \mathsf{ev}_B \cdot (f^C \times C) \cdot (\overline{g} \times C) \\
\equiv & \qquad \{\ \times \text{ functor}\ \} \\
 & f \cdot g \ = \ \mathsf{ev}_B \cdot (f^C \cdot \overline{g} \times C) \\
\equiv & \qquad \{\ \text{exponential universal property (35)}\ \} \\
 & \overline{f \cdot g} \ = \ f^C \cdot \overline{g}
\end{array}$$

□

Notice that the pointwise calculation above can be rephrased, using this result, and taking g as a *point*, *i.e.*, $\overline{g} : \mathbf{1} \longrightarrow A^C$. In this case, $f^C\ \overline{g}$ equals $\overline{f \cdot g}$ as proved above, but now

$\overline{f \cdot g}$ is itself a point of B^C, which corresponds to morphism $f \cdot g$. In other words,

$$f^C \;=\; f \cdot \_$$

35. THE EXPONENTIAL BIFUNCTOR. The exponential functor above can be made into a bifunctor by defining, for each $h : C \longrightarrow D$, an arrow $X^h : X^D \longrightarrow X^C$ as follows:

$$X^h \;\triangleq\; X^D \xrightarrow{\mathsf{sp}} (X^D \times C)^C \xrightarrow{(\mathsf{id}_{X^D} \times h)^C} (X^D \times D)^C \xrightarrow{\mathsf{ev}^C} X^C$$

Note that the exponential bifunctor is *contravariant* in its second argument. Moreover, X^h can be proved equal to post-composition with h, *i.e.*, $X^h = \_ \cdot h$.

36. CARTESIAN CATEGORIES. Categories are classified according to the structure they exhibit. A category with all finite products, or, equivalently, with binary products and final object, is called *Cartesian.*

Observe that the product construction on a category has the structure (up to isomorphism) of an Abelian monoid. To establish notation, let us represent associativity, commutativity and right and left units by the following isomorphisms, natural on A, B and C:

$$\begin{aligned}
\mathsf{a} &: A \times B \times C \longrightarrow A \times (B \times C) \\
\mathsf{s} &: A \times B \longrightarrow B \times A \\
\mathsf{r} &: \mathbf{1} \times A \longrightarrow A \\
\mathsf{l} &: A \times \mathbf{1} \longrightarrow A
\end{aligned}$$

whose inverses are, respectively, a°, s (notice s is its own inverse), r° and l°. In any Cartesian category they are defined as follows:

$$\begin{aligned}
\mathsf{a} &= \langle \pi_1 \cdot \pi_1, \langle \pi_1 \cdot \pi_2, \pi_2 \rangle \rangle \\
\mathsf{s} &= \langle \pi_2, \pi_1 \rangle \\
\mathsf{r}^\circ &= \langle !_A, \mathsf{id}_A \rangle \\
\mathsf{l}^\circ &= \langle \mathsf{id}_A, !_A \rangle
\end{aligned}$$

37. HOUSEKEEPING MORPHISMS. The following morphisms provide a shorthand notation for typical combinations of a and s. We call them *exchange* morphisms as they change the position of some factors in a multiplicative expression. They are particularly useful to handle 'housekeeping' tasks when calculating in a cartesian category.

$$\begin{aligned}
\mathsf{xr} &: A \times B \times C \;\longrightarrow\; A \times C \times B \\
&= \mathsf{a}^\circ \cdot (\mathsf{id} \times \mathsf{s}) \cdot \mathsf{a} \\
\mathsf{xl} &: A \times (B \times C) \;\longrightarrow\; B \times (A \times C) \\
&= \mathsf{a} \cdot (\mathsf{s} \times \mathsf{id}) \cdot \mathsf{a}^\circ \\
\mathsf{m} &: A \times B \times (C \times D) \;\longrightarrow\; A \times C \times (B \times D) \\
&= \mathsf{a} \cdot (\mathsf{xr} \times \mathsf{id}) \cdot \mathsf{a}^\circ \;=\; \mathsf{a}^\circ \cdot (\mathsf{id} \times \mathsf{xl}) \cdot \mathsf{a}
\end{aligned}$$

If the category has finite coproducts, we shall refer to the corresponding associativity, commutativity and unit morphisms as

$$\begin{aligned}\mathsf{a}_+ &: A+B+C \longrightarrow A+(B+C)\\ \mathsf{s}_+ &: A+B \longrightarrow B+A\\ \mathsf{r}_+ &: \emptyset+A \longrightarrow A\\ \mathsf{l}_+ &: A+\emptyset \longrightarrow A\end{aligned}$$

and consider the additive versions of the 'exchange' morphisms:

$$\begin{aligned}\mathsf{xr}_+ &: A+B+C \longrightarrow A+C+B\\ \mathsf{xl}_+ &: A+(B+C) \longrightarrow B+(A+C)\\ \mathsf{m}_+ &: (A+B)+(C+D) \longrightarrow (A+C)+(B+D)\end{aligned}$$

38. CARTESIAN CLOSEDNESS. A Cartesian category in which product has a right adjoint is classified as *Cartesian closed* (or *ccc*, for short). Such a category has exponentials, and therefore the capacity of representing hom-sets as objects and internalising composition. Set is a prime example.

39. DISTRIBUTIVITY. Suppose a category has both finite products and coproducts. If, additionally, binary products distribute over finite coproducts, the category is called *distributive.* Being distributive means there exist two natural isomorphisms

$$\begin{aligned}\mathsf{dl} &: (A+B)\times C \longrightarrow (A\times C)+(B\times C)\\ \mathsf{zl} &: \emptyset\times A \longrightarrow \emptyset\end{aligned}$$

as finite coproducts are generated from binary and nullary coproducts. The latter is, of course, the initial object in the category. 'Right' versions of these isomorphisms are obtained by pre-composing them with s. We shall denote them as $\mathsf{dr} : C\times(A+B) \longrightarrow (C\times A)+(C\times B)$ and $\mathsf{zr} : A\times\emptyset \longrightarrow \emptyset$. Set, as any other *ccc* with coproducts, is distributive. On the other hand, Rel is not. Although distributivity is a much weaker notion than, for instance, cartesian closedness, it has been proposed, notably by R. C. Walters [Wal89], as 'the' natural semantic framework for datatypes and programming. It is also the basic requirement on the semantic category underlying CHARITY References [Coc93] and [CLW93] provide details on this topic.

Let us make a brief incursion on distributivity. First notice that the inverse of dl can be defined easily as

$$\mathsf{dl}^\circ = [\iota_1\times C, \iota_2\times C]$$

or, by application of the exchange law (27), as a *split* involving the same arrows. On the other hand, dl has no pointfree definition in terms of *eithers* or *splits* alone. However, if the category is ccc, it can be defined pointfree as follows:

$$\begin{tikzcd}(A+B)\times C \arrow[r, "{[\iota_1^C\cdot\mathsf{sp},\iota_2^C\cdot\mathsf{sp}]\times C}"] & (A\times C+B\times C)^C\times C \arrow[r, "\mathsf{ev}"] & A\times C+B\times C\end{tikzcd}$$

Let us check the correctness of this definition. This involves two facts $\mathsf{dl}^\circ\cdot\mathsf{dl} = \mathsf{id}$ and $\mathsf{dl}\cdot\mathsf{dl}^\circ = \mathsf{id}$ which will be proved as an exercise in using the categorical 'tool kit' just introduced.

Proof. First verify

$$
\begin{array}{cl}
 & \mathsf{dl}^\circ \cdot \mathsf{dl} \\
= & \qquad \{ \ \mathsf{dl} \text{ definition } \} \\
 & \mathsf{dl}^\circ \cdot \mathsf{ev} \cdot ([\iota_1^C \cdot \mathsf{sp}, \iota_2^C \cdot \mathsf{sp}] \times C) \\
= & \qquad \{ \ \mathsf{ev} \text{ natural } \} \\
 & \mathsf{ev} \cdot ({\mathsf{dl}^\circ}^C \times C) \cdot ([\iota_1^C \cdot \mathsf{sp}, \iota_2^C \cdot \mathsf{sp}] \times C) \\
= & \qquad \{ \ \times \text{ functor } \} \\
 & \mathsf{ev} \cdot ({\mathsf{dl}^\circ}^C \cdot [\iota_1^C \cdot \mathsf{sp}, \iota_2^C \cdot \mathsf{sp}] \times C) \\
= & \qquad \{ \ +\text{-fusion, } \textit{exponential} \text{ functor } \} \\
 & \mathsf{ev} \cdot [(\mathsf{dl}^\circ \cdot \iota_1)^C \cdot \mathsf{sp}, (\mathsf{dl}^\circ \cdot \iota_2)^C \cdot \mathsf{sp}] \times C \\
= & \qquad \{ \ \mathsf{dl}^\circ \text{ definition, } +\text{-cancellation } \} \\
 & \mathsf{ev} \cdot [(\iota_1 \times C)^C \cdot \mathsf{sp}, \iota_2 \times C)^C \cdot \mathsf{sp}] \times C \\
= & \qquad \{ \ \mathsf{sp} \text{ natural } \} \\
 & \mathsf{ev} \cdot [\mathsf{sp} \cdot \iota_1, \mathsf{sp} \cdot \iota_2] \times C \\
= & \qquad \{ \ + \text{ fusion } \} \\
 & \mathsf{ev} \cdot (\mathsf{sp} \cdot [\iota_1, \iota_2] \times C) \\
= & \qquad \{ \ + \text{ reflection } \} \\
 & \mathsf{ev} \cdot (\mathsf{sp} \times C) \\
= & \qquad \{ \ \text{adjunction (31) } \} \\
 & \mathsf{id}_{(A+B)\times C}
\end{array}
$$

and then

$$
\begin{array}{cl}
 & \mathsf{dl} \cdot \mathsf{dl}^\circ \\
= & \qquad \{ \ \mathsf{dl} \text{ and } \mathsf{dl}^\circ \text{ definition} \} \\
 & \mathsf{ev} \cdot ([\iota_1^C \cdot \mathsf{sp}, \iota_2^C \cdot \mathsf{sp}] \times C) \cdot [\iota_1 \times C, \iota_2 \times C] \\
= & \qquad \{ \ + \text{ fusion } \} \\
 & \mathsf{ev} \cdot [([\iota_1^C \cdot \mathsf{sp}, \iota_2^C \cdot \mathsf{sp}] \times C) \cdot (\iota_1 \times C), ([\iota_1^C \cdot \mathsf{sp}, \iota_2^C \cdot \mathsf{sp}] \times C) \cdot (\iota_2 \times C)] \\
= & \qquad \{ \ \times \text{ functor and } + \text{ cancellation } \} \\
 & \mathsf{ev} \cdot [(\iota_1^C \cdot \mathsf{sp}) \times C, (\iota_2^C \cdot \mathsf{sp}) \times C] \\
= & \qquad \{ \ + \text{ fusion } \} \\
 & [\mathsf{ev} \cdot ((\iota_1^C \cdot \mathsf{sp}) \times C), \mathsf{ev} \cdot ((\iota_2^C \cdot \mathsf{sp}) \times C)] \\
= & \qquad \{ \ \times \text{ functor } \} \\
 & [\mathsf{ev} \cdot (\iota_1^C \times C) \cdot (\mathsf{sp} \times C), \mathsf{ev} \cdot (\iota_2^C \times C) \cdot (\mathsf{sp} \times C)] \\
= & \qquad \{ \ \mathsf{ev} \text{ natural and (31) } \} \\
 & [\iota_1, \iota_2] \\
= & \qquad \{ \ + \text{ reflection } \} \\
 & \mathsf{id}_{(A\times C)+(B\times C)}
\end{array}
$$

□

40. Conditionals. In a distributive category conditional expressions can be modelled by coproducts. We adopt the McCarthy conditional constructor written as $(p \rightarrow f,\, g)$, where $p : A \longrightarrow \mathbf{2}$ is a predicate. Intuitively, $(p \rightarrow f,\, g)$ reduces to f if p evaluates to `true` and to g otherwise. The conditional construct is defined as

$$(p \rightarrow f,\, g) \;=\; [f, g] \cdot p?$$

where $p? : A \longrightarrow A + A$ is determined by predicate p as follows

$$p? = \qquad A \xrightarrow{\langle \mathsf{id}, p \rangle} A \times (\mathbf{1} + \mathbf{1}) \xrightarrow{\mathsf{dl}} A \times \mathbf{1} + A \times \mathbf{1} \xrightarrow{\pi_1 + \pi_1} A + A$$

Reference [Gib97] provides a comprehensive set of laws to calculate with conditionals. For example,

$$h \cdot (p \rightarrow f,\, g) \;=\; (p \rightarrow h \cdot f,\, h \cdot g) \tag{41}$$

$$(p \rightarrow f,\, g) \cdot h \;=\; (p \cdot h \rightarrow f \cdot h,\, g \cdot h) \tag{42}$$

$$(p \rightarrow f,\, g) \;=\; (p \rightarrow (p \rightarrow f,\, g),\, (p \rightarrow f,\, g)) \tag{43}$$

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