Revisiting Invariants

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Motivation

Previous work on software components

- as persistent (state-based) and interacting entities
- leads to the development of coalgebraic models as generalised ($=$ parametrized by a strong monad) Mealy machines [Bar00]
- and calculi to compose components and reason compositionally about them [BO02,BO03].

... but somehow neglected the ubiquity of "business rules" in systems design.

Motivation

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... but somehow neglected the ubiquity of "business rules" in systems design.
Clearly, most "business rules" are invariants. But

- how can we calculate with invariants, in a generic way?
- and preserve them along the component assembly process?

Invariants

Definition (by Bart Jacobs)

An invariant for a coalgebra $c : X \to F(X)$ is a predicate $P \subseteq X$ which is "closed under c":

$$x \in P \quad \Rightarrow \quad c(x) \in Pred(F)(P)$$

for all $x \in X$.

Question

Is such a definition amenable to formal calculation?
(formal $\equiv$ in a *let-the-symbols-do-the-work* style)

Modelling vs Calculating

The use of formal modelling methods often raises a kind of

Notation conflict

between

- *descriptiveness* — ie., adequacy to describe domain-specific objects and properties and build suitable models, and
- *compactness* — as required by algebraic reasoning and solution calculation.

Modelling vs Calculating

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between

- *descriptiveness* — ie., adequacy to describe domain-specific objects and properties and build suitable models, and
- *compactness* — as required by algebraic reasoning and solution calculation.

More demanding problems entails the need for a temporary change of the working "mathematical space", e.g.

Laplace transform

From the time-space to the s-space:
$f(t)$ is transformed into $(\mathcal{L}\ f)s = \int_0^\infty e^{-st} f(t)dt$

Quoting Kreyszig's book, p.242

"(...) The Laplace transformation is a method for solving differential equations (...) [which] consists of three main steps:

- 1st step. *The given "hard" problem is transformed into a "simple" equation (subsidiary equation).*
- 2nd step. *The subsidiary equation is solved by* **purely algebraic** *manipulations.*
- 3rd step. *The solution of the subsidiary equation is transformed back to obtain the solution of the given problem.*

In this way the Laplace transformation reduces the problem of solving a differential equation to an **algebraic problem***".*

An "s-space equivalent" for logical quantification

The pointfree ($\mathcal{PF}$) transform

ϕ	$\mathcal{PF}\ \phi$
$\langle \exists\ a\ ::\ b\ R\ a \wedge a\ S\ c \rangle$	$b(R \cdot S)c$
$\langle \forall\ a, b\ :\ b\ R\ a :\ b\ S\ a \rangle$	$R \subseteq S$
$\langle \forall\ a\ ::\ a\ R\ a \rangle$	$id \subseteq R$
$\langle \forall\ x\ :\ x\ R\ b :\ x\ S\ a \rangle$	$b(R \setminus S)a$
$\langle \forall\ c\ :\ b\ R\ c :\ a\ S\ c \rangle$	$a(S\ /\ R)b$
$b\ R\ a \wedge c\ S\ a$	$(b, c)\langle R, S \rangle a$
$b\ R\ a \wedge d\ S\ c$	$(b, d)(R \times S)(a, c)$
$b\ R\ a \wedge b\ S\ a$	$b\ (R \cap S)\ a$
$b\ R\ a \vee b\ S\ a$	$b\ (R \cup S)\ a$
$(f\ b)\ R\ (g\ a)$	$b(f^\circ \cdot R \cdot g)a$
TRUE	$b\ \top\ a$
FALSE	$b\ \bot\ a$

What are R, S, $\bot$, ...?

A transform for logic and set-theory

An old idea

$$\mathcal{PF}(\text{sets, predicates}) \quad = \quad \text{pointfree binary relations}$$

Calculus of binary relations

- 1860 - introduced by De Morgan, embryonic
- 1870 - Peirce finds interesting equational laws
- 1941 - Tarski's school
- 1980's - coreflexive models of sets (Freyd and Scedrov, Eindhoven MPC group and others)

Unifying approach

Everything is a (binary) relation

Binary Relations

Arrow notation

Arrow $B \xleftarrow{R} A$ denotes a binary relation to B (target) from A (source).

Identity of composition

id such that $R \cdot id = id \cdot R = R$

Converse

Converse of R — R° such that $a(R^\circ)b$ iff $b\ R\ a$.

Ordering

"$R \subseteq S$ — the "R is at most S" — the obvious $R \subseteq S$ **ordering**.

Binary Relations

Pointwise meaning

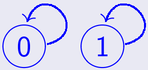
$b \; R \; a$ means that pair $\langle b, a \rangle$ is in R, eg.

$$\begin{array}{ccc} 1 & \leq & 2 \\ \mathtt{John} & \mathit{IsFatherOf} & \mathtt{Mary} \\ 3 & = (1+) & 2 \end{array}$$

Reflexive and coreflexive relations

- Reflexive relation: $id \subseteq R$
- Coreflexive relation: $R \subseteq id$

Sets

Are represented by coreflexives, eg. set $\{0, 1\}$ is 

Algebraic manipulation

Algebraic ("al-djabr") rules, as **Galois connections**

$$f \cdot R \subseteq S \quad \equiv R \subseteq f^\circ \cdot S$$

$$R \cdot f^\circ \subseteq S \quad \equiv R \subseteq S \cdot f$$

$$T \cdot R \subseteq S \quad \equiv R \subseteq T \setminus S$$

or **closure** rules, eg. (for Φ coreflexive),

$$\Phi \cdot R \subseteq S \quad \equiv \Phi \cdot R \subseteq \Phi \cdot S$$

Invariants PF-transformed

Imploding the outermost $\forall$ in Jacobs definition:

$$\langle \forall\ x\ ::\ x \in P \Rightarrow c(x) \in\ Pred(F)(P) \rangle$$

$\equiv$ { sets as coreflexive relations }

$$\langle \forall\ x\ ::\ x\ P\ x \Rightarrow (c\ x)\ Pred(F)(P)\ (c\ x) \rangle$$

$\equiv$ { PF-transform rule $(f\ \ b)R(g\ \ a) \equiv b(f^\circ \cdot R \cdot g)a$ }

$$\langle \forall\ x\ ::\ x\ P\ x \Rightarrow x(c^\circ \cdot Pred(F)(P) \cdot c)x) \rangle$$

$\equiv$ { drop variables (PF-transform of inclusion) }

$$P \subseteq c^\circ \cdot Pred(F)(P) \cdot c$$

$\equiv$ { introduce relator ; shunting rule }

$$c \cdot P \subseteq (\mathsf{F}\ P) \cdot c$$

$\equiv$ { introduce Reynolds combinator }

$$c(\mathsf{F}\ P \leftarrow P)c$$

About Reynolds arrow

Reynolds arrow combinator is a relation on functions

$$f(R \leftarrow S)g \quad \equiv \quad f \cdot S \subseteq R \cdot g$$

cf. diagram

$$\begin{tikzcd} B \arrow[d, "f"'] & A \arrow[l, "S"'] \arrow[d, "g"] \arrow[dl, phantom, "\subseteq"] \\ C & D \arrow[l, "R"] \end{tikzcd}$$

useful in expressing properties of functions — namely **monotonicity**

$$B \xleftarrow{f} A \text{ is monotonic} \quad \equiv \quad f(\leq_B \leftarrow \leq_A)f$$

polymorphism (free theorem):

$$\mathsf{G}\,A \xleftarrow{f} \mathsf{F}\,A \text{ is polymorphic} \quad \equiv \quad \langle \forall\ R\ ::\ f(\mathsf{G}\,R \leftarrow \mathsf{F}\,R)f \rangle$$

etc

Invariants as coreflexive bisimulations

Re-working the calculation backwards, and considering two coalgebras c and d and a relation R on their state spaces:

$$
\begin{array}{ll}
& c(\mathsf{F}\,R \leftarrow R)d \\
\equiv & \quad \{ \text{ Reynolds combinator } \} \\
& c \cdot R \subseteq (\mathsf{F}\,R) \cdot d \\
\equiv & \quad \{ \text{ shunting rule; drop variables (PF-transform of inclusion) } \} \\
& \langle \forall\; x, y \;::\; x\; R\; y \Rightarrow x(c^{\circ} \cdot \mathsf{F}\,R \cdot d)y) \rangle \\
\equiv & \quad \{ \text{ PF-transform rule } (f\;\; b)R(g\;\; a) \equiv b(f^{\circ} \cdot R \cdot g)a \; \} \\
& \langle \forall\; x, y \;::\; x\; R\; y \Rightarrow (c\;\; x)\; \mathsf{F}\,R\; (d\;\; y) \rangle
\end{array}
$$

Invariants as coreflexive bisimulations

... arrive at:

Definition (by Bart Jacobs):

A bisimulation for coalgebras $c : X \to F(X)$ and $d : Y \to F(Y)$ is a relation $R \subseteq X \times Y$ which is "closed under c and d":

$$(x, y) \in R \quad \Rightarrow \quad (c(x), d(y)) \in Rel(F)(R)$$

for all $x \in X$ and $y \in Y$.

Question

Having put both *invariants* and *bisimulations* in a common setting

— as Reynolds arrows —

how can our *reasoning power* be enriched?

Why Reynolds arrow matters?

Useful and manageable PF-properties

For example

$$
\begin{aligned}
id \leftarrow id &= id && (1)\\
(R \leftarrow S)^\circ &= R^\circ \leftarrow S^\circ && (2)\\
R \leftarrow S \subseteq V \leftarrow U &\Leftarrow R \subseteq V \wedge U \subseteq S && (3)\\
(R \leftarrow V) \cdot (S \leftarrow U) &\subseteq (R \cdot S) \leftarrow (V \cdot U) && (4)
\end{aligned}
$$

recalled from Backhouse's "*On a relation on functions*" (1990)

Why Reynolds arrow matters

Get monotony on the consequent side and thus,

$$S \leftarrow R \quad \subseteq \quad (S \cup V) \leftarrow R \tag{5}$$

$$\top \leftarrow S \quad = \quad \top \tag{6}$$

anti-monotony on the antecedent one

$$R \leftarrow \bot \quad = \quad \top \tag{7}$$

and two distributive laws:

$$S \leftarrow (R_1 \cup R_2) \quad = \quad (S \leftarrow R_1) \cap (S \leftarrow R_2) \tag{8}$$

$$(S_1 \cap S_2) \leftarrow R \quad = \quad (S_1 \leftarrow R) \cap (S_2 \leftarrow R) \tag{9}$$

Why Reynolds arrow matters

Ex: *id* is a bisimulation

$$
\begin{array}{ll}
 & c(\mathsf{F}\, id \leftarrow id)d \\
\equiv & \quad \{ \text{ relator F preserves the identity } \} \\
 & c(id \leftarrow id)d \\
\equiv & \quad \{ \ (1) \ \} \\
 & c\ (id)\ d \\
\equiv & \quad \{ \ id\ x = x \ \} \\
 & c = d
\end{array}
$$

Why Reynolds arrow matters

Ex: the converse of a bisimulation is a bisimulation

$$
\begin{array}{ll}
 & c(\mathsf{F}\,R \leftarrow R)d \\
\equiv & \quad \{ \ \text{converse} \ \} \\
 & d(\mathsf{F}\,R \leftarrow R)^{\circ}c \\
\equiv & \quad \{ \ (2) \ \} \\
 & d((\mathsf{F}\,R)^{\circ} \leftarrow R^{\circ})c \\
\equiv & \quad \{ \ \text{relator } \mathsf{F} \ \} \\
 & d(\mathsf{F}(R^{\circ}) \leftarrow R^{\circ})c
\end{array}
$$

Why Reynolds arrow matters

Ex: bisimulations are closed under union

Therefore,

$$
\begin{array}{cl}
 & (\mathsf{F}\,R_1 \leftarrow R_1) \cap (\mathsf{F}\,R_2 \leftarrow R_2) \\
\subseteq & \{ \ (5)\ (\text{twice})\ ;\ \text{monotonicity of meet} \ \} \\
 & ((\mathsf{F}\,R_1 \cup \mathsf{F}\,R_2) \leftarrow R_1) \cap ((\mathsf{F}\,R_1 \cup \mathsf{F}\,R_2) \leftarrow R_2) \\
= & \{ \ (8) \ \} \\
 & (\mathsf{F}\,R_1 \cup \mathsf{F}\,R_2) \leftarrow (R_1 \cup R_2) \\
= & \{ \ \text{relators} \ \} \\
 & \mathsf{F}(R_1 \cup R_2) \leftarrow (R_1 \cup R_2)
\end{array}
$$

Why Reynolds arrow matters

Ex: behavioural equivalence is a bisimulation

$$u R v \;\equiv\; [\![c]\!] u = [\![d]\!] v \qquad R \text{ is a bisimulation}$$

$$
\begin{array}{cl}
 & c(\mathsf{F}\,([\![c]\!]^\circ \cdot [\![d]\!]) \leftarrow [\![c]\!]^\circ \cdot [\![d]\!])d \\
\equiv & \qquad \{ \text{ definition } \} \\
 & [\![c]\!]^\circ \cdot [\![d]\!] \;\subseteq\; c^\circ \cdot \mathsf{F}\,([\![c]\!]^\circ \cdot [\![d]\!]) \cdot d \\
\equiv & \qquad \{ \text{ relators } \} \\
 & [\![c]\!]^\circ \cdot [\![d]\!] \;\subseteq\; c^\circ \cdot \mathsf{F}\,[\![c]\!]^\circ \cdot \mathsf{F}\,[\![d]\!] \cdot d \\
\equiv & \qquad \{ \text{ converse } \} \\
 & [\![c]\!]^\circ \cdot [\![d]\!] \;\subseteq\; (\mathsf{F}\,[\![c]\!] \cdot c)^\circ \cdot \mathsf{F}\,[\![d]\!] \cdot d
\end{array}
$$

Why Reynolds arrow matters

Ex: behavioural equivalence is a bisimulation

$$
\begin{array}{ll}
 & [\![c]\!]^\circ \cdot [\![d]\!] \subseteq (\mathsf{F}\, [\![c]\!] \cdot c)^\circ \cdot \mathsf{F}\, [\![d]\!] \cdot d \\
\equiv & \qquad \{ \text{ universal property of coinductive extension } \} \\
 & [\![c]\!]^\circ \cdot [\![d]\!] \subseteq (\omega \cdot [\![c]\!])^\circ \cdot \omega \cdot [\![d]\!] \\
\equiv & \qquad \{ \text{ converse } \} \\
 & [\![c]\!]^\circ \cdot [\![d]\!] \subseteq [\![c]\!]^\circ \cdot \omega^\circ \cdot \omega \cdot [\![d]\!] \\
\equiv & \qquad \{ \text{ Lambek (final coalgebra is an isomorphism) } \} \\
 & \mathit{true}
\end{array}
$$

Why Reynolds arrow matters

... too simple and obvious, even *without* Reynolds arrow in the play. But, consider now the equivalence between Jacobs and Aczel-Mendler's definition of *bisimulation*

Definition (by Aczel & Mendler)

Given two coalgebras $c : X \longrightarrow F(X)$ and $d : Y \longrightarrow F(Y)$ an F-bisimulation is a relation $R \subseteq X \times Y$ which can be extended to a coalgebra ρ such that projections π_1 and π_2 lift to F-comorphisms, as expressed by

$$
\begin{tikzcd}
 & R \arrow[dl, "\pi_1"'] \arrow[d, "\rho"'] \arrow[dr, "\pi_2"] & \\
X \arrow[d, "c"'] & \mathsf{F}\,R \arrow[dl, "\mathsf{F}\,\pi_1"'] \arrow[dr, "\mathsf{F}\,\pi_2"] & Y \arrow[d, "d"] \\
\mathsf{F}\,X & & \mathsf{F}\,Y
\end{tikzcd}
$$

Jacobs $\equiv$ Aczel & Mendler

$$
\begin{array}{ll}
 & c(\mathsf{F}\,R \leftarrow R)d \\
\equiv & \qquad \{ \text{ tabulate } R = \pi_1 \cdot \pi_2^{\circ} \ \} \\
 & c(\mathsf{F}(\pi_1 \cdot \pi_2^{\circ}) \leftarrow (\pi_1 \cdot \pi_2^{\circ}))d \\
\equiv & \qquad \{ \text{ relator commutes with composition and converse } \} \\
 & c(((\mathsf{F}\,\pi_1) \cdot (\mathsf{F}\,\pi_2)^{\circ}) \leftarrow (\pi_1 \cdot \pi_2^{\circ}))d \\
\equiv & \qquad \{ \text{ fusion [CIC'06] law } \} \\
 & c((\mathsf{F}\,\pi_1 \leftarrow \pi_1) \cdot ((\mathsf{F}\,\pi_2)^{\circ} \leftarrow \pi_2^{\circ}))d \\
\equiv & \qquad \{ \ (2) \ \} \\
 & c((\mathsf{F}\,\pi_1 \leftarrow \pi_1) \cdot (\mathsf{F}\,\pi_2 \leftarrow \pi_2)^{\circ})d \\
\equiv & \qquad \{ \text{ go pointwise (composition) } \} \\
 & \langle \exists\, a \ :: \ c(\mathsf{F}\,\pi_1 \leftarrow \pi_1)a \wedge d(\mathsf{F}\,\pi_2 \leftarrow \pi_2)a \rangle
\end{array}
$$

cf.

$$
\begin{tikzcd}
X \arrow[ddd, "c"'] & & Y \arrow[ll, "R"'] \arrow[ddd, "d"] \\
 & Z \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[d, "a"] & \\
 & \mathsf{F}\,Z \arrow[dl, "\mathsf{F}\,\pi_1"'] \arrow[dr, "\mathsf{F}\,\pi_2"] & \\
\mathsf{F}\,X & & \mathsf{F}\,Y \arrow[ll, "\mathsf{F}\,R"]
\end{tikzcd}
$$

Why Reynolds arrow matters

Meaning of $\langle \exists\ a\ ::\ c(\mathsf{F}\,\pi_1 \leftarrow \pi_1)a \wedge d(\mathsf{F}\,\pi_2 \leftarrow \pi_2)a \rangle$:

> there exists a coalgebra a whose carrier is the "graph" of bisimulation R and which is such that projections π_1 and π_2 lift to the corresponding coalgebra morphisms.

Comments:

- One-slide-long proofs are easier to grasp — for a (longer) bi-implication proof of the above see Backhouse & Hoogendijk's paper on *dialgebras* (1999)
- Rule $(r \cdot s^\circ) \leftarrow (f \cdot g^\circ) = (r \leftarrow f) \cdot (s \leftarrow g)^\circ$ does most of the work — its proof is an example of generic, stepwise PF-reasoning [CIC'06, paper to appear]

Why Reynolds arrow matters

Ex: coalgebra morphisms entail bisimulation

Immediate, since inclusion of functions is equality:

$$c(\mathsf{F}\,h \leftarrow h)d \quad \equiv \quad c \cdot h = (\mathsf{F}\,h) \cdot d \tag{10}$$

However, in the Aczel & Mendler setting becomes:

Let $h : d \longleftarrow c$ a coalgebra morphism and conjecture $\gamma : \mathsf{F}\,h \longleftarrow h$

$$\gamma \;=\; \mathsf{F}\,(\pi_2)^{\circ} \cdot d \cdot \pi_2 \tag{11}$$

Now prove the diagram commutes: i.e., both π_1 and π_2 are coalgebra morphisms, i.e.,

$$\mathsf{F}\,\pi_1.\gamma = c \cdot \pi_1 \qquad \mathsf{F}\,\pi_2.\gamma = d \cdot \pi_2 \tag{12}$$

Clearly, π_2 is a coalgebra *iso*morphism. Then, prove that π_1 is also a colagebra morphism, i.e.,

$$c \cdot \pi_1 \;=\; \mathsf{F}\,\pi_1 \cdot \gamma \tag{13}$$

Why Reynolds arrow matters

$$
\begin{array}{ll}
& c \cdot \pi_1 = \mathsf{F}\,\pi_1 \cdot \gamma \\
\equiv & \qquad \{ \text{ conjecture on } \gamma\text{; functors } \} \\
& c \cdot \pi_1 = \mathsf{F}\,(\pi_1 \cdot (\pi_2)^\circ) \cdot d \cdot \pi_2 \\
\equiv & \qquad \{ \ h = \pi_1 \cdot (\pi_2)^\circ \ \} \\
& c \cdot \pi_1 = \mathsf{F}\,h \cdot d \cdot \pi_2 \\
\equiv & \qquad \{ \ h \text{ morphism } \} \\
& c \cdot \pi_1 = c \cdot h \cdot \pi_2 \\
\equiv & \qquad \{ \ \pi_2 \text{ iso, } h = \pi_1 \cdot (\pi_2)^\circ \ \} \\
& c \cdot \pi_1 = c \cdot \pi_1
\end{array}
$$

Why Reynolds arrow matters

Now the converse direction: if h is a function st the diagram commutes, h is a coalgebra morphism.

$$
\begin{array}{cl}
& c \cdot h \;=\; \mathsf{F}\, h \cdot d \\
\equiv & \qquad \{\; h = \pi_1 \cdot (\pi_2)^\circ, \text{ functors } \} \\
& c \cdot \pi_1 \cdot (\pi_2)^\circ \;=\; \mathsf{F}\, \pi_1 \cdot \mathsf{F}\, (\pi_2)^\circ \cdot d \\
\equiv & \qquad \{\; \text{hyp: } (12) \;\} \\
& \mathsf{F}\, \pi_1 . \gamma \cdot (\pi_2)^\circ \;=\; \mathsf{F}\, \pi_1 \cdot \mathsf{F}\, (\pi_2)^\circ \cdot d \\
\equiv & \qquad \{\; \gamma \text{ definition and } \pi_2 \text{ is iso } \} \\
& \mathsf{F}\, \pi_1 . \gamma \;=\; \mathsf{F}\, \pi_1 \cdot \gamma
\end{array}
$$

Invariants

Invariants are coreflexive bisimulations

$$c(\mathsf{F}\,\Phi \longleftarrow \Phi)c$$

Get for free:

- *id* (everywhere true predicate) is largest invariant
- $\perp$ (everywhere false) is the least one
- Invariants are closed by disjunction (ie. union), ...

Modalities

Invariants bring about *modalities*:

$$
\begin{array}{rcl}
c(\mathsf{F}\,\Phi \leftarrow \Phi)c & \equiv & c \cdot \Phi \subseteq \mathsf{F}\,\Phi \cdot c \\
& \equiv & \{ \text{ shunting rule } \} \\
& & \Phi \subseteq \underbrace{c^{\circ} \cdot (\mathsf{F}\,\Phi) \cdot c}_{\bigcirc_c \Phi}
\end{array}
$$

since we define the *"next time X holds"* modal operator as

$$
\bigcirc_c X \stackrel{\mathrm{def}}{=} c^{\circ} \cdot (\mathsf{F}\,X) \cdot c
$$

Φ invariant $\equiv$ $\Phi \subseteq \bigcirc\Phi$

$$
\begin{array}{rcl}
c(\mathsf{F}\,\Phi \leftarrow \Phi)c & \equiv & c \cdot \Phi \subseteq \mathsf{F}\,\Phi \cdot c \\
& \equiv & \Phi \subseteq c^{\circ} \cdot \mathsf{F}\,\Phi \cdot c \\
& \equiv & \Phi \subseteq \bigcirc\Phi
\end{array}
$$

Modalities

In PF-refactoring of *database theory* [Oli06] has derived Galois connection

$$\pi_{g,f} R \subseteq S \quad \equiv \quad R \subseteq g^{\circ} \cdot S \cdot f \tag{14}$$

in order to get (for free) properties of lower adjoint $\pi_{g,f}$.

Interesting enough, an instance of such a connection

$$\pi_c \Phi \subseteq \Psi \quad \equiv \quad \Phi \subseteq \bigcirc_c \Psi \tag{15}$$

(within coreflexives) can be re-used to obtain (again for free) properties — now — of the upper adjoint $\bigcirc_c$:

Modalities

As as upper adjoint in a Galois connection,

- $\bigcirc_c$ is **monotonic** — thus simple proofs such as

$$
\begin{array}{cl}
 & \Phi \text{ is an invariant} \\
\equiv & \quad \{ \text{ PF-definition of invariant } \} \\
 & \Phi \subseteq \bigcirc_c \Phi \\
\Rightarrow & \quad \{ \text{ monotonicity } \} \\
 & \bigcirc_c \Phi \subseteq \bigcirc_c (\bigcirc_c \Phi) \\
\equiv & \quad \{ \text{ PF-definition of invariant } \} \\
 & \bigcirc_c \Phi \text{ is an invariant}
\end{array}
$$

- $\bigcirc_c$ **distributes** over conjunction, that is PF-equality

$$\bigcirc_c (\Phi \cdot \Psi) \quad = \quad (\bigcirc_c \Phi) \cdot (\bigcirc_c \Psi)$$

holds, etc

Modalities

Further modal operators, for instance $\Box\Psi$ — *henceforth* Ψ — usually defined as *the largest invariant at most* Ψ:

$$\Box\Psi = \langle \bigcup \Phi :: \Phi \subseteq \Psi \cap \bigcirc_c \Phi \rangle$$

which shrinks to a greatest (post)fix-point

$$\Box\Psi = \langle \nu \Phi :: \Psi \cdot \bigcirc_c \Phi \rangle$$

where meet (of coreflexives) is replaced by composition, as this paves the way to agile reasoning

Modalities

Ex: $\Box\Phi = \Phi \;\equiv\; \Phi\ inv$

(cf, [Jacobs,06] Lemma 4.2.6, pg 109)

$\Box\Phi \subseteq \Phi$ is obvious from the definition, but

$$
\begin{array}{ll}
& \Phi\ inv \\
\equiv & \qquad \{ \text{ just proved } \} \\
& \Phi \subseteq \bigcirc\Phi \\
\equiv & \qquad \{ \ \Phi\cdot \text{ monotonic; composition of coreflexives is involutive } \} \\
& \Phi \subseteq \Phi \cdot \bigcirc\Phi \\
\Rightarrow & \qquad \{ \text{ greatest fixed point induction: } x \leq fx \;\Rightarrow\; x \leq \nu f \ \} \\
& \Phi \subseteq \Box\Phi
\end{array}
$$

Modalities

$$
\begin{array}{ll}
 & \Phi \subseteq \Box\Phi \\
\Rightarrow & \quad \{ \ \Box\Phi \subseteq f(\Box\Phi) \text{ for } fx = \Phi \cdot \bigcirc x \text{ and gfp induction: } \nu_f \leq f\nu_f \ \} \\
 & \Phi \subseteq \Phi \cdot \bigcirc(\Box\Phi) \\
\equiv & \quad \{ \ \text{shunting of coreflexives} \ \} \\
 & \Phi \subseteq \bigcirc(\Box\Phi) \\
\Rightarrow & \quad \{ \ \text{monotony; } \Box\Phi \subseteq \Phi \ \} \\
 & \Phi \subseteq \bigcirc\Phi \\
\equiv & \quad \{ \ \text{definition} \ \} \\
 & \Phi \ \mathit{inv}
\end{array}
$$

Modalities

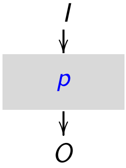
Ex: $\Box\Phi \;\subseteq\; \Box\Box\Phi$

$$
\begin{array}{ll}
 & \Box\Phi \;\subseteq\; \Box\Box\Phi \\
\equiv & \qquad \{ \text{ definition } \} \\
 & \Box\Phi \;\subseteq\; (\nu_X :: \Box\Phi \cdot \bigcirc X) \\
\Leftarrow & \qquad \{ \text{ gfp induction } \} \\
 & \Box\Phi \;\subseteq\; \Box\Phi \cdot \bigcirc(\Box\Phi) \\
\equiv & \qquad \{ \; \Box\Phi \cdot \Phi = \Box\Phi \text{ because } \cap \text{ is composition and } \Box\Phi \subseteq \Phi \; \} \\
 & \Box\Phi \;\subseteq\; \Box\Phi \cdot \Phi \cdot \bigcirc(\Box\Phi) \\
\equiv & \qquad \{ \text{ shunting of coreflexives and } \nu_f \leq f\,\nu_f \; \} \\
 & \Box\Phi \;\subseteq\; \Phi \cdot \bigcirc(\Box\Phi) \quad \equiv \quad \mathit{true}
\end{array}
$$

Recall: Components as coalgebras

A (generic) component p with input interface I and output interface O

$$p : O \longleftarrow I$$



is a pair

$$\langle u_p \in U_p, \overline{a}_p : \mathsf{B}(U_p \times O)^I \longleftarrow U_p \rangle$$

where

- point u_p is the ‘initial’ or ‘seed’ state.
- B is an arbitrary **strong** monad.

Recall: Components as coalgebras

The semantics of p is the behaviour produced by starting at initial state u_p and **unfolding** over coalgebra $\overline{a}_p$:

$$[\![p]\!] \quad = \quad [\!(\overline{a}_p)\!]\, u_p$$

$$\begin{tikzcd}
\mathsf{B}(\nu \times O)^I & \nu \arrow[l, "\omega"'] \\
\mathsf{B}(U_p \times O)^I \arrow[u, "\mathsf{B}([\!(\overline{a}_p)\!] \times O)^I"] & U_p \arrow[l, "\overline{a}_p"] \arrow[u, "{[\!(\overline{a}_p)\!]}"']
\end{tikzcd}$$

That is, an action will not simply produce an output and a continuation state, but a B -structure of such pairs.

Monad B's **unit** (η) and **multiplication** (μ) provide, respectively, a value embedding and a 'flatten' operation to unravel nested behavioural annotations.

Invariants as types

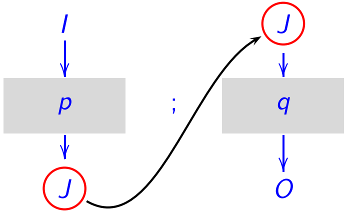
- Each (elementary) component is an aggregation of methods over a shared state space, typically restricted by an (often complex) invariant,
- whose underlying mathematical space can be organised as a category whose
  - objects are coreflexives (representing invariants)
  - arrows

$$f : \Psi \longleftarrow \Phi \;\equiv\; f(\Psi \leftarrow \Phi)f \;\equiv\; f \cdot \Phi \subseteq \Psi \cdot f$$

- Current work: on the structure of this category [paper in preparation]

Combinators preserve invariants

Ex: Pipeline $p\,;q$:



$$a_{p;q} : \mathsf{B}(U_p \times U_q \times O) \longleftarrow U_p \times U_q \times I$$

$$\begin{aligned}
a_{p;q} = {} & U_p \times U_q \times I \xrightarrow{\ \cong\ } U_p \times I \times U_q \xrightarrow{a_p \times \mathrm{id}} \mathsf{B}(U_p \times K) \times U_q \\
& \xrightarrow{\ \tau_r\ } \mathsf{B}(U_p \times K \times U_q) \xrightarrow{\ \cong\ } \mathsf{B}(U_p \times (U_q \times K)) \\
& \xrightarrow{\mathsf{B}(\mathrm{id} \times a_q)} \mathsf{B}(U_p \times \mathsf{B}(U_q \times O)) \xrightarrow{\ \mathsf{B}\tau_l\ } \mathsf{BB}(U_p \times (U_q \times O)) \\
& \xrightarrow{\ \cong\ } \mathsf{BB}(U_p \times U_q \times O) \xrightarrow{\ \mu\ } \mathsf{B}(U_p \times U_q \times O)
\end{aligned}$$

Combinators preserve invariants

Invariants are preserved $\equiv$ the following is a well-typed arrow:

$$
\begin{aligned}
a_{p;q} &= \Phi_p \times \Phi_q \times I \xrightarrow{\cong} \Phi_p \times I \times \Phi_q \xrightarrow{a_p \times \mathrm{id}} \mathrm{B}(\Phi_p \times K) \times \Phi_q \\
&\xrightarrow{\tau_r} \mathrm{B}(\Phi_p \times K \times \Phi_q) \xrightarrow{\cong} \mathrm{B}(\Phi_p \times (\Phi_q \times K)) \\
&\xrightarrow{\mathrm{B}(\mathrm{id} \times a_q)} \mathrm{B}(\Phi_p \times \mathrm{B}(U_q \times O)) \xrightarrow{\mathrm{B}\tau_l} \mathrm{BB}(\Phi_p \times (\Phi_q \times O)) \\
&\xrightarrow{\cong} \mathrm{BB}(\Phi_p \times \Phi_q \times O) \xrightarrow{\mu} \mathrm{B}(\Phi_p \times \Phi_q \times O)
\end{aligned}
$$

Combinators preserve invariants

which is an immediate consequence of the (generic) way in which combinators are defined:

- natural transformations are trivial: each polymorphic construction α verifies $\alpha(S \leftarrow R)\alpha$ for all R, S.
- functorial arrows:

$$
\begin{array}{ll}
& \mathsf{F}\, f(\mathsf{F}\,\Phi \leftarrow \mathsf{F}\,\Psi)\mathsf{F}\, f \\
\equiv & \quad \{ \text{ Reynolds combinator } \} \\
& \mathsf{F}\, f \cdot \mathsf{F}\,\Psi \subseteq \mathsf{F}\,\Phi \cdot \mathsf{F}\, f \\
\equiv & \quad \{ \text{ functors } \} \\
& \mathsf{F}\,(f \cdot \Phi) \subseteq \mathsf{F}\,(\Psi \cdot f) \\
\Leftarrow & \quad \{ \text{ monotonicity } \} \\
& f(\Phi \leftarrow \Psi)f
\end{array}
$$

- component actions which, by hypothesis, preserve their own invariants

Summary

Such conceptual tools are applicable at different design levels:

- micro: synthesising component invariants from the individual methods over complex data strucutures
  (cf, Necco & Oliveira & Visser, *Extended Static Checking by Rewriting Pointfree Relations*, 2007
  and Oliveira, *Reinvigorating pen-and-paper proofs in VDM: the pointfree approach*, 2006)
- macro: invariant preservation in the component calculus.
- architectural: global (non structural) "bussiness rules" over components' aggregations

Summary

- Rôle of PF-patterns: clear-cut expression of complex logic structures once expressed in less symbols
- Stress the syntactic aspect of formal reasoning, a kind of "let-the-symbols-do-the-work" style of calculation, often neglected by too much emphasis on domain-specific, semantic concerns.
- Rôle of PF-patterns: much easier to spot synergies among different theories

In particular, a synergy between a relational construct, traditionally employed in explaining and reasoning about parametric polymorphism, and the coalgebraic approach to bisimulations and invariants emerged.