On Refinements of Algebraic Specifications

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Aims

In view of software reuse, it would be interesting:

- transpose the traditional assumption of the preservation of observability on the observational stepwise refinement process;
- Consider an alternative of the refinement concept based on general maps (and not just on signature morphisms)- refinements via interpretations.

Outline

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The algebraic specification process

In general algebraic assumption:

- programs are algebras;
- computations are terms;

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To specify a software system:

- Define a (multi-sorted) an adequate signature
- Express the system requirements in a logical system;

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- programs are algebras;
- computations are terms;

To specify a software system:

- Define a (multi-sorted) an adequate signature
- Express the system requirements in a logical system;

Algebraic Specification: $SP = (\Sigma, Mod(SP))$
Development task: Choose a Σ-algebra from $Mod(SP)$ to implement the system;

The software development - the stepwise refinement methodology

Definition (Refinement)

Let SP and SP′ be algebraic specifications. SP′ is a refinement of SP, ($SP \rightsquigarrow SP'$), if:

1. $Sig(SP) = Sig(SP')$;
2. $Mod(SP') \subseteq Mod(SP)$;

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Stepwise Refinement Process:

$$SP_0 \rightsquigarrow SP_1 \rightsquigarrow SP_2 \rightsquigarrow \cdots \rightsquigarrow SP_{n-1} \rightsquigarrow SP_n,$$

Vertical composition: $SP \rightsquigarrow SP'$ and $SP' \rightsquigarrow SP''$ then $SP \rightsquigarrow SP''$:
$Sig(SP) = Sig(SP') = Sig(SP'')$ and
$Mod(SP'') \subseteq Mod(SP') \subseteq Mod(SP)$

Example of refinement

Spec Cell =

$[Sorts]$ $\quad elt; cell;$

$[OP]$ $\quad put : elt, cell \longrightarrow cell; get : cell \longrightarrow elt;$

$[AX]$ $\quad (\forall e : elt)(\forall c : cell) get(put(e, c)) = e;$

Spec $CELL^*$= enrich CELL by

$[AX]$ $\quad (\forall e, e' : elt)(\forall c : cell) put(e, put(e', c))) = put(e, c)$

$$CELL \rightsquigarrow CELL^*$$

The stepwise refinement methodology

Lemma (Satisfaction lemma-[GB92])

Let $\sigma : \Sigma \to \Sigma'$ *be a signature morphism,* ϕ *a* Σ*-equation and* A' *a* Σ'*-algebra. Then* $A' \models \sigma(\phi)$ *iff* $A' \restriction_\sigma \models \phi$.

The stepwise refinement methodology

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Definition (σ-refinement)

$$SP \rightsquigarrow_\sigma SP'$$

if

$$Mod(SP') \restriction_\sigma \subseteq Mod(SP)$$

where $Mod(SP') \restriction_\sigma = \{A' \restriction_\sigma | A' \in Mod(SP')\}$.

The stepwise refinement methodology

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Definition (σ-refinement)

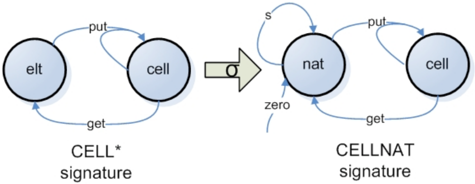
$$SP \leadsto_{\sigma} SP'$$
if
$$Mod(SP') \restriction_{\sigma} \subseteq Mod(SP)$$

where $Mod(SP') \restriction_{\sigma} = \{A' \restriction_{\sigma} | A' \in Mod(SP')\}$.

vertical composition: $SP \leadsto_{\sigma} SP' \leadsto_{\phi} SP''$
$Mod(SP'') \restriction_{\phi \circ \sigma} \subseteq Mod(SP') \restriction_{\sigma} \subseteq Mod(SP)$

Example of σ-refinement

$$
\begin{array}{llll}
\sigma : & Sig(CELL) & \rightarrow & Sig(CELLNAT) \\
 & elt & \rightarrow & nat \\
 & cell & \rightarrow & cell \\
 & get & \rightarrow & get \\
 & put & \rightarrow & put \\
 & & \rightarrow & s \\
 & & \rightarrow & zero
\end{array}
$$

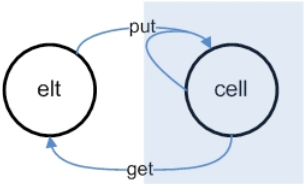


$$CELL \leadsto_{\sigma} CELLNAT$$

Encapsulated data and observability

In presence of encapsulated data:

- Observational equality (between elements);
- Observational equivalence (between models);



Observational equality

Definition (Observable context)

Let $\Sigma = (S, \Omega)$ *be a signature,* $Obs \subseteq S$ *a set of observable sorts,* X *a variable set for* Σ *and* $Z = (\{z_s\})_{s \in S}$ *a* S*-sorted set of singulares sets. A* s*-context over* Σ *is a term* $c \in T_\Sigma(X \cup \{z_s\})_{s'}$, $s' \in Obs$.

Definition (Observational equality)

For $a, b \in A_s$, $a \approx^A_{Obs} b$ *if* $\forall c \in \mathcal{C}^{Obs}_\Sigma(s)\ \forall \alpha, \beta : X \cup \{z_s\} \to A$ *such that* $\alpha(x) = \beta(x)$ *for all* $x \in X$ *and* $\alpha(z_s) = a$ *and* $\beta(z_s) = b \Rightarrow I_\alpha(c) = I_\beta(c)$.

Observational equality

Definition (Observable context)

Let $\Sigma = (S, \Omega)$ be a signature, $Obs \subseteq S$ a set of observable sorts, X a variable set for Σ and $Z = (\{z_s\})_{s \in S}$ a S-sorted set of singulares sets. A s-context over Σ is a term $c \in T_\Sigma(X \cup \{z_s\})_{s'}$, $s' \in Obs$.

Definition (Observational equality)

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- $\models_{\approx_{Obs}}$-observational satisfaction
- $A/\approx^A_{Obs}$-observational behaviour
- behavioural operator: $Mod($**behaviour** SP **wrt** $\approx_{Obs}) = \{A \in Alg(Sig(SP)) | A/\approx^A_{Obs} \in Mod(SP)\}$

Observational equality

Theorem (eg. [BH96])

$$A \models_{\approx_{Obs}} \phi \textit{ iff } A/\approx^{A}_{Obs} \models \phi$$

Fact

For $SP = \langle \Sigma, \Phi \rangle$:

$$Mod(\textbf{behaviour } SP \textbf{ wrt } \approx_{Obs}) = \{A \in Alg(\Sigma) | A/\approx^{A}_{Obs} \in SP\}$$

$$= \{A \in Alg(\Sigma) | A/\approx^{A}_{Obs} \models \Phi\} = \{A \in Alg(\Sigma) | A \models_{\approx_{Obs}} \Phi\}$$

Encapsulated data and observability

Spec $Cell^* =$

$[Sorts]$ $elt; cell;$

$[OP]$ $put : elt, cell \to cell; get : cell \to elt;$

$[AX]$ $(\forall e : elt)(\forall c : cell) get(put(e, c)) = e;$

$(\forall e, e' : elt)(\forall c : cell) put(e, put(e', c))) = put(e, c);$

$B_{elt} = \mathbb{N}$

$B_{cell} = \mathbb{N}^*$

$put^B(m, c) = mc$

$get^B(\epsilon) = 0$

$get^B(nc) = n$ for $n, m \in \mathbb{N}, c \in \mathbb{N}^*$

$$put^B(e, put^B(e', c)) = put^B(e, e'c) = ee'c \text{ and } put^B(e, c) = ec \text{ but,}$$

$$get^B(put^B(e, put^B(e', c))) = get(put^B(e, c))$$

$$get^B(put(e'', (put^B(e, put^B(e', c))))) = get^B(put(e'', (put^B(e, c)))$$

$$\vdots$$

Then, $put^B(e, put^B(e', c)) \approx^B_{Obs} put^B(e, c)$

"Traditional" observational stepwise refinement process

Definition (Observational σ-refinement)

Let SP and SP′ be algebraic specifications, Obs ⊆ S and $\sigma : Sig(SP) \to Sig(SP')$.

$$SP \rightsquigarrow_{\sigma}^{\approx Obs} SP'$$

if

$$\textbf{behaviour } SP \textbf{ wrt } \approx_{Obs \rightsquigarrow_\sigma} SP',$$

ie., when

$$Mod(SP') \restriction_\sigma \subseteq Mod(\textbf{behaviour } SP \textbf{ wrt } \approx_{Obs})$$

$$SP_1 \rightsquigarrow_{\sigma_1}^{\approx Obs} SP_2 \rightsquigarrow_{\sigma_2}^{\approx \sigma_1(Obs)} \ldots SP_{n-1} \rightsquigarrow_{\sigma_n}^{\approx \sigma_n \circ \cdots \circ \sigma_1(Obs)} SP_n \Rightarrow SP_1 \rightsquigarrow_{\sigma_n \circ \cdots \circ \sigma_1}^{\approx Obs} SP_n$$

Outline

Motivations

Changing of *Obs* set in observational stepwise refinement process:

- According to *O.O. Paradigm*, only the input/output data should be desencapsulated;
- For security reasons may be important encapsulate same data sorts;
- For software verification tasks;
- ...

The vertical composition of observational refinements

Fact ([Hen97])

Let

$$\begin{array}{ccccc} \Sigma & & \Sigma & & \Sigma \\ SP & \rightsquigarrow^{\approx_{Obs}} & SP' & \rightsquigarrow^{\approx_{Obs'}} & SP'' \\ A & & A & & \\ \approx^A_{Obs} & & \approx^A_{Obs'} & & \end{array}$$

Then, $\approx^A_{Obs} \geq \approx^A_{Obs'} \Rightarrow SP \rightsquigarrow^{\approx_{Obs}} SP''$

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Fact (Generalization)

Let

$$\begin{array}{ccccc} \Sigma & & \Sigma' & & \Sigma'' \\ SP & \rightsquigarrow^{\approx_{Obs}}_{\sigma} & SP' & \rightsquigarrow^{\approx_{Obs'}}_{\phi} & SP'' \\ A' \restriction_{\sigma} & & A' & & \\ \approx^{A' \restriction_{\sigma}}_{Obs} & & \approx^{A'}_{Obs'} & & \end{array}$$

Then, $\approx^{A' \restriction_{\sigma}}_{Obs} \geq (\approx^{A'}_{Obs'}) \restriction_{\sigma} \Rightarrow SP \rightsquigarrow^{\approx_{Obs}}_{\phi \circ \sigma} SP''$

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Then, $\approx^{A' \upharpoonright_{\sigma}}_{Obs} \geq (\approx^{A'}_{Obs'}) \upharpoonright_{\sigma} \Rightarrow SP \rightsquigarrow^{\approx_{Obs}}_{\phi \circ \sigma} SP''$

Intuitive idea: for $\sigma = id$, $Obs \subseteq Obs' \Rightarrow \mathcal{C}^{Obs}_{\Sigma} \subseteq \mathcal{C}^{Obs'}_{\Sigma} \Rightarrow \approx_{Obs} \geq \approx_{Obs'}$

The vertical composition of observational refinements

Definition (Observational morphism)

Let $\Sigma = (S, \Omega)$ *and* $\Sigma' = (S', \Omega')$ *signatures,* Obs *and* Obs' *sets of observable sorts for* Σ *and* Σ' *respectively, and* $\sigma : \Sigma \to \Sigma'$ *a signature morphism.* σ *is an* $Obs - Obs'$*-observational morphism if for any* $s \in Obs, \sigma(s) \in Obs'$*;*

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Theorem

Let $\sigma : \Sigma \to \Sigma'$ *be an* $Obs - Obs'$*-observational morphism and* A *be a* Σ'*-algebra. Then* $(\approx_{Obs',A'}) \restriction_\sigma \leq \approx_{Obs,(A' \restriction_\sigma)}$ *.*

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Corollary

Let σ *be an* $Obs - Obs'$*-observable morphism and* ϕ *be an* $Obs' - Obs''$*-observable morphism. If* $SP \rightsquigarrow_{\sigma}^{\approx_{Obs}} SP'$ *and* $SP' \rightsquigarrow_{\phi}^{\approx_{Obs'}} SP''$*, then* $SP \rightsquigarrow_{\phi \circ \sigma}^{\approx_{Obs}} SP''$*.*

Concerning to the equational case:

Let ϕ be a Σ-equation and A be a Σ-algebra and $SP = \langle \Sigma, \Phi \rangle$ be an equational specification:

- $A \models_{\approx_{Obs}} \phi \Rightarrow A \models_{\approx_{Obs \setminus \{s\}}} \phi$
- $Mod(\textbf{behavioural } SP \textbf{ wrt } \approx_{Obs}) \subseteq Mod(\textbf{behavioural } SP \textbf{ wrt } \approx_{Obs \setminus \{s\}})$
- $SP \rightsquigarrow^{\approx_{Obs}} SP'$ implies $SP \rightsquigarrow^{\approx_{Obs \setminus \{s\}}} SP'$

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- $Mod($**behavioural** SP **wrt** $\approx_{Obs}) \subseteq Mod($**behavioural** SP **wrt** $\approx_{Obs \setminus \{s\}})$
- $SP \rightsquigarrow^{\approx_{Obs}} SP'$ implies $SP \rightsquigarrow^{\approx_{Obs \setminus \{s\}}} SP'$

However, in the general case:

- $Mod($**behavioural** SP **wrt** $\approx_{Obs \setminus \{s\}}) \not\subseteq Mod($**behavioural** SP **wrt** $\approx_{Obs})$
- $SP \rightsquigarrow^{\approx_{Obs}} SP'$ does not implies $SP \rightsquigarrow^{\approx_{Obs \cup \{s\}}} SP'$

Desencapsulation in equational specifications

Lemma

Let $\Sigma = (S, \Omega)$ be a signature, Obs a set of observable sorts for Σ and Φ be a set of Σ-equations. Then, for any Σ-algebra A and for any $s \in Obs$,

$$A \models_{\approx_{Obs \cup \{s\}}} \Phi \text{ iff } \left(A \models_{\approx_{Obs}} \Phi \text{ and } A \models \Phi'\right)$$

where

$$\Phi' = \Phi_s \cup \{c(t) = c(t') \mid t \approx t' \in \Phi_h, h \in S \setminus (Obs \cup \{s\}), c \in \mathcal{C}_{\Sigma}^{\{s\}}(h)\}.$$

Notation

$$SP_s = \langle \Sigma, \Phi' \rangle$$

Desencapsulation in equational specifications

$$SP \rightsquigarrow^{\approx_{Obs}} SP'$$

$$Mod(SP') \subseteq Mod(\textbf{behaviour } SP \textbf{ wrt } \approx_{Obs})$$

$$Mod(SP') \cap Mod(SP_s) \subseteq Mod(\textbf{behaviour } SP \textbf{ wrt } \approx_{Obs}) \cap Mod(SP_s)$$

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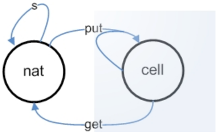
Theorem

Let Φ be a set of Σ-equations and $SP = \langle \Sigma, \Phi \rangle$ and SP' be two specifications such that $SP \rightsquigarrow^{\approx_{Obs}} SP'$. Then

$$SP \rightsquigarrow^{\approx_{Obs \cup \{s\}}} SP' \cap SP_s.$$



Spec $CELL^{*} =$

$(\forall e : elt)(\forall c : cell) get(put(e, c)) \approx e;$

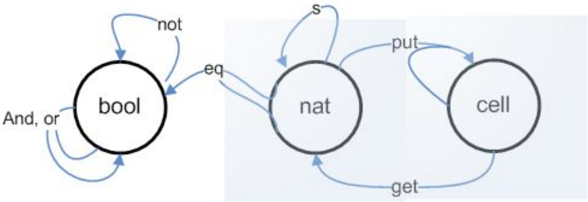
$(\forall e, e' : elt)(\forall c : cell) put(e, put(e', c))) \approx put(e, c);$

$$\Phi' = \Phi_s \cup \{c(t) \approx c(t') \mid t \approx t' \in \Phi_h, h \in S \setminus (Obs \cup \{s\}), c \in \mathcal{C}_{\Sigma}^{\{s\}}(h)\}.$$

Spec $CELL^{*}_{cell} =$

$(\forall e, e' : elt)(\forall c : cell) put(e, put(e', c))) \approx put(e, c);$

- For any SP such that $CELL^{*} \rightsquigarrow^{\approx_{elt}} SP$, we have that $CELL^{*} \rightsquigarrow^{\approx_{elt \cup cell}} SP + CELL^{*}_{cell}$



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Spec $CELLNATEQ_{nat}$ =

$(\forall x : nat).s(p(x)) \approx x$;

$(\forall e, e' : nat)(\forall c : cell).get(put(e, put(e', c))) \approx get(put(e, c))$;

$(\forall e, e', e'' : nat)(\forall c : cell).get(put(e'', put(e, put(e', c))))$
$\approx get(put(e'', put(e, c)))$;

⋮

Outline

The refinement by interpretation concept - motivations

The previous refinement formalizations are based on signatures morphisms:

- the formulas structure is preserved, ie.:
  $\sigma(f(a_1, \ldots, a_n)) = \sigma(f)(\sigma(a_1) \ldots \sigma(a_n))$;
- a formula is mapped into another one;
- the choice of other maps to translate specifications can be useful in the view of the software reuse.

Definitions and notation

- conditional equation: is a pair $\langle \Gamma, e \rangle$, for $\{e\} \cup \Gamma \subseteq_{fin} \mathrm{Eq}_{\Sigma}(X)$;
- translation from Σ to Σ': is a globally finite *multi-function* $\tau : \mathrm{Eq}_{\Sigma}(X) \rightarrow \mathrm{Eq}_{\Sigma'}(X')$

$\tau^* : Ceq_{\Sigma}(X) \rightarrow Ceq_{\Sigma'}(X')$

for any $\xi = \langle \Gamma, e \rangle \in Ceq_{\Sigma}(X)$,

$$\tau^*(\xi) = \{\langle \bigcup_{t \approx t' \in \Gamma} \tau(t \approx t'), e' \rangle : e' \in \tau(e)\}.$$

Interpretations

Definition (Interpretation)

Let $\tau : \mathrm{Eq}_{\Sigma}(X) \to \mathrm{Eq}_{\Sigma'}(X')$. Let SP be a specification over Σ. We say that τ interprets SP if there is a specification SP′ over Σ' such that, for any $\xi \in \mathrm{Ceq}_{\Sigma}(X)$,

$$SP \models \xi \text{ if and only if } SP' \models \tau(\xi).$$

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$$SP \models \xi \text{ if and only if } SP' \models \tau(\xi).$$

Definition (τ-model)

Let $\tau : \mathrm{Eq}(X)_\Sigma \to \mathrm{Eq}(X)_{\Sigma'}$. *A* Σ'*-algebra* A' *is a* τ*-model of SP if for any* $\xi \in Fm(\Sigma)$, $SP \models \xi$ *implies* $A' \models \tau(\xi)$. *We define also* $SP^\tau = \langle \Sigma', Mod(SP^\tau) \rangle$ *where* $Mod(SP^\tau)$ *is the classe of* τ*-models of SP.*

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Definition (τ-model)

Let $\tau : \mathrm{Eq}(X)_{\Sigma} \to \mathrm{Eq}(X)_{\Sigma'}$. A Σ'-algebra A' is a τ-model of SP if for any $\xi \in Fm(\Sigma)$, $SP \models \xi$ implies $A' \models \tau(\xi)$. We define also $SP^{\tau} = \langle \Sigma', Mod(SP^{\tau}) \rangle$ where $Mod(SP^{\tau})$ is the classe of τ-models of SP.

Theorem

If τ interprets SP, then the specification SP^{τ} is the τ-interpretation of SP with the largest class of models.

Example of interpretation

Lemma ([BR03])

For $\mathcal{L}$ *the propositional language and* $\{\phi_1 \approx \psi_1, \ldots, \phi_n \approx \psi_n\}, \{\phi \approx \psi\} \subseteq \mathrm{Eq}_{\mathcal{L}}(X)$*:*

$$\mathit{BOOL} \models \langle\{\phi_1 \approx \psi_1, \ldots, \phi_n \approx \psi_n\}, \{\phi \approx \psi\}\rangle$$

iff

$$\mathit{HEYTING} \models \langle\{\neg\neg\phi_1 \approx \neg\neg\psi_1, \ldots, \neg\neg\phi_n \approx \neg\neg\psi_n\}, \{\neg\neg\phi \approx \neg\neg\psi\}\rangle$$

Therefore, the translation $\tau(t \approx t') = \{\neg\neg t \approx \neg\neg t'\}$ interprets BOOL on HEYTING.

Axiomatization of SP^{τ}

Proposition ([Cze01])

Let $\Sigma = (S, \Omega)$ *be a signature and* $\tau : \mathrm{Eq}(X)_{\Sigma} \to \mathrm{Eq}(X)_{\Sigma}$. *Then, TFAE:*

1. τ *commutes with substitutions, i.e., for any* $e \in \mathrm{Eq}_{\Sigma}(X)$, $\sigma(\tau(e)) = \tau(\sigma(e))$;
2. *There exists a* S*-sorted set of equations* $\Delta(x, y) \subseteq \mathrm{Eq}_{\Sigma}(X)$ *such that, for any* $t \approx t' \in \mathrm{Eq}_{\Sigma}(X)_s$, $\tau(t \approx t') = \Delta_s(t, t')$.

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Theorem

Let $\tau : \mathrm{Eq}(X)_\Sigma \to \mathrm{Eq}(X)_\Sigma$ *and* $SP = \langle \Sigma, \Phi \rangle$. *If* τ *interprets* SP *and commutes with arbitrary substitutions then* $SP^\tau = \langle \Sigma', \tau(\Phi) \rangle$. *Moreover, if* Φ *is finite then* SP^τ *is finitely axiomatized.*

Refinement via interpretations

Definition (Refinement via interpretation)

Let SP be a specification over Σ *and* $\tau : \mathrm{Eq}(X)_\Sigma \to \mathrm{Eq}(X')_{\Sigma'}$ *a translations which interprets SP.* $SP \rightharpoondown_\tau SP'$*,if for any* $\xi \in \mathrm{Ceq}_\Sigma(X)$,

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Theorem

Let SP be a specifications over Σ *and* $\tau : \mathrm{Eq}(X)_\Sigma \to \mathrm{Eq}(X')_{\Sigma'}$ *a translations which interprets SP. Then, for every* SP' *specification over* Σ',

$$SP^\tau \rightsquigarrow SP' \text{ implies } SP \rightharpoonup_\tau SP'.$$

Refinement via interpretations

Definition (Refinement via interpretation)

Let SP be a specification over Σ *and* $\tau : \mathrm{Eq}(X)_\Sigma \to \mathrm{Eq}(X')_{\Sigma'}$ *a translations which interprets SP.* $SP \rightharpoondown_\tau SP'$*,if for any* $\xi \in \mathrm{Ceq}_\Sigma(X)$,

$$SP \models \xi \textit{ implies } SP' \models \tau(\xi).$$

Theorem

Let SP be a specifications over Σ *and* $\tau : \mathrm{Eq}(X)_\Sigma \to \mathrm{Eq}(X')_{\Sigma'}$ *a translations which interprets SP. Then, for every* SP' *specification over* Σ',

$$SP^\tau \rightsquigarrow SP' \textit{ implies } SP \rightharpoondown_\tau SP'.$$

Corollary

Let $\tau : \mathrm{Eq}(X)_\Sigma \to \mathrm{Eq}(X')_{\Sigma'}$ *which interprets* $SP = \langle \Sigma, \Phi \rangle$*. Then,*

$$SP' \models \tau(\Phi) \textit{ we have } SP \rightharpoondown_\tau SP'.$$

Example: Equality test

Spec Nat=

[*Sorts*] nat;

[*Op*] $s : nat \rightarrow nat$;

[*Ax*] $s(x) \approx s(y) \Rightarrow x \approx y$

Spec NatEq= enrich BOOL by

[*Sorts*] nat;

[*Op*] $s : nat \rightarrow nat; eq : nat, nat \rightarrow bool$;

[*Ax*] $eq(x, x) \approx true$

$eq(x, y) \approx true \Rightarrow eq(y, x) \approx true$;

$eq(x, y) \approx true \wedge eq(y, z) \approx true \Rightarrow eq(x, z) \approx true$;

$eq(x, y) \approx true \Rightarrow eq(s(x), s(y)) \approx true$;

$eq(s(x), s(y)) \approx true \Rightarrow eq(x, y) \approx true$;

Example: Equality test

Spec Nat=

[Sorts] $nat;$

[Op] $s : nat \to nat;$

[Ax] $s(x) \approx s(y) \Rightarrow x \approx y$

Spec NatEq= enrich BOOL by

[Sorts] $nat;$

[Op] $s : nat \to nat; eq : nat, nat \to bool;$

[Ax] $eq(x, x) \approx true$

$eq(x, y) \approx true \Rightarrow eq(y, x) \approx true;$

$eq(x, y) \approx true \wedge eq(y, z) \approx true \Rightarrow eq(x, z) \approx true;$

$eq(x, y) \approx true \Rightarrow eq(s(x), s(y)) \approx true;$

$eq(s(x), s(y)) \approx true \Rightarrow eq(x, y) \approx true;$

Considering:

$$\tau(x : nat \approx y : nat) = \{eq(x : nat, y : nat) \approx true\}$$

$NatEq \models eq(s(x), s(y)) \approx true \Rightarrow eq(x, y) \approx true$

$$Nat \rightharpoonup_\tau NatEq$$

Outline

It would be interesting:

- create a calculus for the refinements via interpretation;
- integrate the refinements via interpretations on the traditional stepwise refinement process;
- explore, in this perspective, other results from algebraic theory of deductive systems and from the theory of conservative translations.

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