1 The Relational Calculus

1.1 Basic structure

$$(R \cdot S) \cdot T \;=\; R \cdot (S \cdot T) \tag{1}$$
$$R \;=\; R \cdot id \;=\; id \cdot R \tag{2}$$

$$R \;\subseteq\; R \tag{3}$$
$$R \subseteq S \wedge S \subseteq T \;\Rightarrow\; R \subseteq T \tag{4}$$
$$R \subseteq S \wedge S \subseteq R \;\Rightarrow\; R = S \tag{5}$$

$$S \subseteq T \wedge R \subseteq U \;\Rightarrow\; S \cdot R \subseteq T \cdot U \tag{6}$$

1.2 Relational taxonomy

R reflexive:	id	$\subseteq$	R
R coreflexive:	R	$\subseteq$	id
R transitive:	$R \cdot R$	$\subseteq$	R
R symmetric:	R	$\subseteq$	R°
R anti-symmetric:	$R \cap R^\circ$	$\subseteq$	id
R connected:	$R \cup R^\circ$	$\subseteq$	$\top$
R entire:	id	$\subseteq$	$\mathsf{ker}\, R$
R simple:	$\mathsf{img}\, R$	$\subseteq$	id
R surjective:	R°		entire
R injective:	R°		simple

1.3 PF-transformation rules

"Guardanapo":

$$b(f^\circ \cdot R \cdot g)a \;\equiv\; (f\ b)R(g\ a) \tag{7}$$

Left-division:

$$b\,(R \setminus Y)\,a \;\equiv\; \langle \forall\, c \;:\; c\,R\,b :\; c\,Y\,a \rangle \tag{8}$$

Pointwise ordering on functions:

$$f \mathrel{\dot{\sqsubseteq}} g \;\equiv\; f \subseteq {\sqsubseteq} \cdot g \;\equiv\; \langle \forall\, a \;::\; (f\ a) \sqsubseteq (g\ a) \rangle \tag{9}$$

1.4 Table of useful Galois connections

$$(f\ X) \subseteq Y \equiv X \subseteq (g\ Y)$$

Description	$f = g^\flat$	$g = f^\sharp$	**Obs.**
converse	$(\_)^\circ$	$(\_)^\circ$	
shunting 1	$(h\cdot)$	$(h^\circ\cdot)$	NB: h is a function
shunting 2	$(\cdot h^\circ)$	$(\cdot h)$	NB: h is a function
ldivision	$(R\cdot)$	$(R\setminus\ )$	R under …
rdivision	$(\cdot R)$	$(\ /R)$	… over R
division	$(R/)$	$(\ \setminus R)$	
range	ρ	$(\cdot\top)$	lower $\subseteq$ restricted to coreflexives
domain	δ	$(\top\cdot)$	lower $\subseteq$ restricted to coreflexives
implication	$(R\cap\ )$	$(R\Rightarrow\ )$	Note that $(R\Rightarrow) = (\neg R\cup\ )$
difference	$(\_ - R)$	$(R\cup\ )$	
PROPERTIES			
cancellation	$X \subseteq (g \cdot f)X$	$(f \cdot g)Y \subseteq Y$	
definition	$f\ X = \bigcap\{Y \mid X \subseteq gY\}$	$g\ Y = \bigcup\{X \mid f\ X \subseteq Y\}$	
distribution	$f(X \cup Y) = (f\ X) \cup (f\ Y)$	$g(X \cap Y) = (g\ X) \cap (g\ Y)$	$f(\bigcup_i X_i) = \bigcup_i (f\ X_i)$ $g(\bigcap_i X_i) = \bigcap_i (g\ X_i)$

1.5 Other Galois connections

Meet-universal

$$X \subseteq (R \cap S) \;\equiv\; (X \subseteq R) \wedge (X \subseteq S) \tag{10}$$

Join-universal

$$(R \cup S) \subseteq X \;\equiv\; (R \subseteq X) \wedge (S \subseteq X) \tag{11}$$

Split-universal

$$X \subseteq \langle R, S\rangle \quad \equiv \quad \pi_1 \cdot X \subseteq R \;\wedge\; \pi_2 \cdot X \subseteq S \tag{12}$$

Either-universal

$$X = [R\,, S] \quad \equiv \quad X \cdot i_1 = R \quad \wedge \quad X \cdot i_2 = S \tag{13}$$

1.6 "Almost" Galois connections

"Shunting" rules for S a simple relation:

$$\begin{array}{rcl} S \cdot R \subseteq T & \equiv & (\delta\, S) \cdot R \;\subseteq\; S^\circ \cdot T \quad (14)\\ R \cdot S^\circ \subseteq T & \equiv & R \cdot \delta\, S \;\subseteq\; T \cdot S \quad (15)\end{array}$$

Variants concerning domain and range:

$$\begin{array}{rcl} \delta\, R \subseteq X & \equiv & R \subseteq R \cdot X \quad (16)\\ \rho\, R \subseteq X & \equiv & R \subseteq X \cdot R \quad (17)\end{array}$$

1.7 Converses

$$\begin{array}{rcl} R \subseteq R^\circ & \equiv & R = R^\circ \quad (18)\\ R & \subseteq & R \cdot R^\circ \cdot R \quad (19)\\ R^{\circ\circ} & = & R \quad (20)\\ (R \cdot S)^\circ & = & S^\circ \cdot R^\circ \quad (21)\\ (R \cap S)^\circ & = & R^\circ \cap S^\circ \quad (22)\\ & & \quad (23)\end{array}$$

1.8 Dedekind axiom

Alternative formulations: left modular law, right modular law, weak distributivity:

$$\begin{array}{rcl} (R \cdot S) \cap T & \subseteq & R \cdot (S \cap (R^\circ \cdot T)) \quad (24)\\ (R \cdot S) \cap T & \subseteq & (R \cap (T \cdot S^\circ)) \cdot S \quad (25)\\ (R \cdot S) \cap T & \subseteq & (R \cap (T \cdot S^\circ)) \cdot (S \cap (R^\circ \cdot T)) \quad (26)\end{array}$$

Useful instances (for $T = id$, $R = R^\circ$ and for $S = id$):

$$\begin{array}{rcl} (R^\circ \cdot S) \cap id & \subseteq & (R \cap S)^\circ \cdot (R \cap S) \quad (27)\\ R \cap T & \subseteq & R \cdot (id \cap (R^\circ \cdot T)) \quad (28)\end{array}$$

1.9 Coreflexives

$$\begin{array}{rcl} \Phi & = & \Phi^\circ \;=\; \Phi \cdot \Phi \;=\; \Phi \cap id \quad (29)\\ \Phi \cdot \Psi & = & \Phi \cap \Psi \quad (30)\\ \Phi \cdot R \subseteq S & \equiv & \Phi \cdot R \subseteq \Phi \cdot S \quad (31)\\ R \cdot \Phi \subseteq S & \equiv & R \cdot \Phi \subseteq S \cdot \Phi \quad (32)\end{array}$$

1.10 Relational divisions

$$\begin{array}{rcl} (R \setminus S) \cdot f & = & R \setminus (S \cdot f) \quad (33)\\ R \setminus (S \setminus T) & = & (S \cdot R) \setminus T \quad (34)\\ R \,/\, (S \cup T) & = & (R \,/\, S) \cap (R \,/\, T) \quad (35)\\ (S \cup T) \setminus R & = & (S \setminus R) \cap (T \setminus R) \quad (36)\end{array}$$

1.11 Meets

$$\begin{array}{rcl} R \cdot (S \cap T) = (R \cdot S) \cap (R \cdot T) & \Leftarrow & (\mathsf{ker}\, R) \cdot T \subseteq T \vee (\mathsf{ker}\, R) \cdot S \subseteq S \quad (37)\\ (S \cap T) \cdot R = (S \cdot R) \cap (T \cdot R) & \Leftarrow & T \cdot \mathsf{img}\, R \subseteq T \vee S \cdot \mathsf{img}\, R \subseteq S \quad (38)\\ (S \cap T) \cdot f & = & (S \cdot f) \cap (T \cdot f) \quad (39)\end{array}$$

$$\begin{array}{rcl} (R \cap S) \cap T & \equiv & R \cap (S \cap T) \quad (40)\\ R \cap S & \equiv & S \cap R \quad (41)\\ R \cap R & \equiv & R \quad (42)\\ R \subseteq S & \equiv & R = R \cap S \quad (43)\\ R \cap S \subseteq R & \wedge & R \cap S \subseteq S \quad (44)\\ T \cdot (R \cap S) & \subseteq & T \cdot R \cap T \cdot S \quad (45)\\ (R \cap S) \cdot T & \subseteq & R \cdot T \cap S \cdot T \quad (46)\end{array}$$

1.12 Kernel and image

$$\begin{array}{rcl} \mathsf{ker}\, R & = & R^\circ \cdot R \quad (47)\\ R \subseteq S & \Rightarrow & \mathsf{ker}\, R \subseteq \mathsf{ker}\, S \quad (48)\\ (\mathsf{ker}\, R)^\circ & = & \mathsf{ker}\, R \quad (49)\\ R & \subseteq & R \cdot \mathsf{ker}\, R \quad (50)\\ \mathsf{ker}\,(R \cdot S) & = & S^\circ \cdot \mathsf{ker}\, R \cdot S \quad (51)\\ (R^\circ \cdot S) \cap id & = & \mathsf{ker}\,(R \cap id) \quad (52)\end{array}$$

$$\mathsf{img}\, R \;=\; \mathsf{ker}\, R^\circ \tag{53}$$

1.13 Domain and range

$$
\begin{aligned}
\delta\, R &= \ker R \cap id && (54)\\
R &= R \cdot \delta\, R && (55)\\
\delta\,(R \cdot S) &= \delta\,(\delta\, R \cdot S) && (56)\\
\delta\,(R \cap S) &= (R^\circ \cdot S) \cap id && (57)
\end{aligned}
$$

$$
\rho\, R = \delta\, R^\circ \qquad (58)
$$

1.14 Splits

Definition equivalent to (12)

$$
\langle R, S\rangle = \pi_1^\circ \cdot R \cap \pi_2^\circ \cdot S \qquad (59)
$$

The same definition pointwise: for all a, b, c

$$
(a,b)\langle R, S\rangle c \equiv a\, R\, c \wedge b\, S\, c \qquad (60)
$$

Split cancellation

$$
\pi_1 \cdot \langle R, S\rangle = R \cdot \delta\, S \quad \wedge \quad \pi_2 \cdot \langle R, S\rangle = S \cdot \delta\, R \qquad (61)
$$

Split (conditional) fusion:

$$
\langle R, S\rangle \cdot T = \langle R \cdot T, S \cdot T\rangle \;\Leftarrow\; R \cdot (\mathsf{img}\, T) \subseteq R \vee S \cdot (\mathsf{img}\, T) \subseteq S \qquad (62)
$$

Split absorption

$$
\langle R \cdot T, S \cdot U\rangle = (R \times S) \cdot \langle T, U\rangle \qquad (63)
$$

Splits and converses:

$$
\langle R, S\rangle^\circ \cdot \langle X, Y\rangle = (R^\circ \cdot X) \cap (S^\circ \cdot Y) \qquad (64)
$$

Therefore:

$$
\ker \langle R, S\rangle = \ker R \cap \ker S \qquad (65)
$$

1.15 Eithers

Definition:

$$
\begin{aligned}
[R\,, S] &= (R \cdot i_1^\circ) \cup (S \cdot i_2^\circ) && (66)\\
R + S &= [i_1 \cdot R\,, i_2 \cdot S] && (67)
\end{aligned}
$$

From (13), all coproduct properties extend to relations, in particular:

$$
\begin{aligned}
id &= [i_1\,, i_2] && (68)\\
[R\,, S] \cdot i_1 = R \;&\wedge\; [R\,, S] \cdot i_2 = S && (69)\\
T \cdot [R\,, S] &= [T \cdot R\,, T \cdot S] && (70)\\
[T\,, P] \cdot (R + S) &= [T \cdot R\,, P \cdot S] && (71)
\end{aligned}
$$

Eithers and converses:

$$
\begin{aligned}
[R\,, S] \cdot [T\,, U]^\circ &= (R \cdot T^\circ) \cup (S \cdot U^\circ) && (72)\\
(R + S)^\circ &= R^\circ + S^\circ && (73)
\end{aligned}
$$

1.16 Relational projection

Definition

$$
\pi_{g,f} R \stackrel{\mathrm{def}}{=} g \cdot R \cdot f^\circ \qquad (74)
$$

Property

$$
\pi_{g,f} R \subseteq S \equiv g(S \leftarrow R)f \qquad (75)
$$

1.17 Equality

Ping-pong rule:

$$
R = S \equiv R \subseteq S \wedge S \subseteq R \qquad (76)
$$

Indirection:

$$
\begin{aligned}
R = S &\equiv \langle \forall\, X :: (X \subseteq R \wedge X \subseteq S)\rangle && (77)\\
R = S &\equiv \langle \forall\, X :: (R \subseteq X \wedge S \subseteq X)\rangle && (78)
\end{aligned}
$$