Towards a Linear Algebra of Programming

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Motivation

Formal methods are going quantitative — further to predicting "**it may happen**" one wants to know "**how often it will happen**"

As happened with physics in the past, computer science is becoming **probabilistic**

Probability theory particularly relevant in **security** analysis of information flow

Haskell library **PFP** for probabilistic functional programming — distribution monad, etc (Erwig and Kollmansberger, 2006)

A new probability perspective

However, traditional notation for probabilities is too **descriptive** and not meant for calculation as we understand it today.

Quoting Hehner (2011) — also wg2.1#64:

> *Perhaps a thousand years ago the philosophers of the time [might give] reasons why their answer is right. Now we* **don't argue***; we formalize,* **calculate***, and unformalize.*

From a functional programming perspective, one may identify

probabilistic functions

as something half way between **relations** and traditional **functions**: they express the propensity, or likelihood of ambiguous (or "bad") outputs.

Example: fault-injected multiplication

Safe multiplication (over $\mathbb{N}_0$):

$$(a*) = \textbf{for}\ (a+)\ 0$$

that is,

$$a * 0 = 0$$
$$a * (n + 1) = a + a * n$$

Bad multiplication, **fault-injected** — 5% probability of a wrong base case (in extended functional notation):

$$a * 0 =_{.95} 0$$
$$a * 0 =_{.05} a$$
$$a * (n + 1) =_1 a + a * n$$

Example: fault-injected multiplication

The same in Haskell's PFP library (Erwig and Kollmansberger, 2006):

```
a * 0 = D [(0,0.95),(a,0.05)]
a * (n+1) = do x <- a * n
               return $ a + x
```

monadic over distributions:

```
newtype Dist a = D {unD :: [(a,ProbRep)]}

instance Monad Dist where
  return x = D [(x,1)]
  d >>= f  = D [(y,q*p) | (x,p) <- unD d, (y,q) <- unD (f x)]
  fail _   = D []
```

Question: does the fault in the base case carry over to the overall function? In what extent? (**Quantify** fault propagation.)

Non-deterministic functions

Nondeterministic outputs — set-valued functions are relations

$$f = \Lambda R \quad \Leftrightarrow \quad \langle \forall\ b, a\ ::\ b\ R\ a \Leftrightarrow b \in f\ a \rangle \tag{1}$$

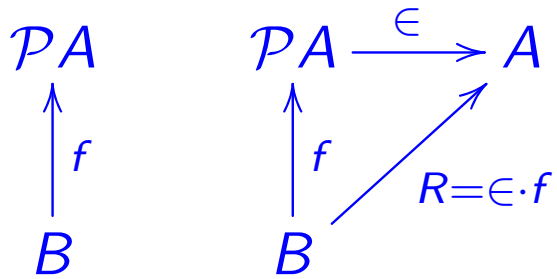
that is,

$$A \to \mathcal{P} B \underset{\Lambda}{\overset{(\in \cdot)}{\cong}} A \to B \tag{2}$$

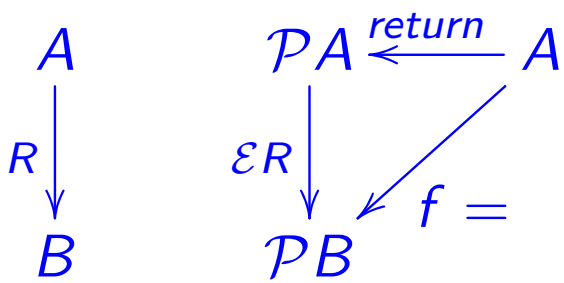
where $A \to B$ on the right hand side is the **relational type** $A \to B$ of all relations $R \subseteq B \times A$.

Nondeterministic functions

An adjunction, offering two ways for reasoning — one relational (**Rel**)



the other monadic (**Set**):



$$\mathcal{E}R \cdot return$$

$$where\ (\mathcal{E}R)s = \{b | a \leftarrow s; bRa\}$$

The same choice in going probabilistic — **monadic**? **what else**?

Probabilistic functions

Outputs become distributions,

$$A \to \mathcal{D}B \quad \cong \quad A \to B \tag{3}$$

where $\mathcal{D}B$ is the B-distribution data type

$$\mathcal{D}B \quad = \quad \{\mu \in [0,1]^B \mid \sum_{b \in B} \mu\ b = 1\} \tag{4}$$

and where $[0,1]$ is the interval of all non-negative reals at most 1.

However, what does $A \to B$ on the right hand side of (3) mean?

Probabilistic functions

One has:

$$
\begin{array}{cc}
 & A \to [0,1]^B \\
\Leftrightarrow & \{ \text{ uncurrying } \} \\
 & A \times B \to [0,1] \\
\Leftrightarrow & \{ \text{ swapping } \} \\
 & B \times A \to [0,1]
\end{array}
$$

where $B \times A \to [0,1]$ can be identified with the set of all **matrices** taking elements from $[0,1]$ with as many **columns** (resp. **rows**) as elements in A (resp. B).

Left stochastic matrices

In fact:

$$A \to \mathcal{D}B \quad \cong \quad A \to_{LS} B \tag{5}$$

where LS denotes the **category** of **left-stochastic** matrices (columns in such matrices add up to 1).

Such a **matrix**-transform is captured by the universal property, for all $f :: A \to \mathcal{D}B$ and LS-matrix M:

$$M = [\![f]\!] \quad \Leftrightarrow \quad \langle \forall\ b, a\ ::\ M(b, a) = (f\ a)b \rangle \tag{6}$$

Research question:

> *Is LS "as useful" to probabilistic reasoning as Rel is to non-deterministic reasoning in the AoP (Bird and de Moor, 1997) ?*

Towards a LAoP

My answer:

I believe so.

But — many things to be explained:

- **categories of matrices** — what's this?
- category of **LS matrices** — what's this?
- the **AoP** is pointfree — universal property (6) above is pointwise...

Answering these questions will generalize the **AoP** into something one may identify as a **Linear** Algebra of Programming (**LAoP**).

Linear algebra for software verification

Could not agree more on...

> *"(...) our key idea is to adopt linear algebra as the lingua franca of software verification"*

quoted from

> *LAP: Linear Algebra of bounded resources Programs*

— a project of SQIG at the Telecommunications Institute (IT) in Lisbon (`http://sqig.math.ist.utl.pt/work/LAP`).

However — **old-fashioned** matrix calculus needs to be **spruced up**... **;-)**

Uups!

"Using matrix notation such a set of simultaneous equations takes the form $A \cdot x = b$ *where* x *is the vector of unknown values,* A *is the matrix of coefficients and* b *is the vector of values on the right side of the equation."*

"In this way a set of equations has been reduced to a **single** *equation."*

"This is a tremendous improvement in **concision** *that does not incur any loss of* **precision!***"*

Roland Backhouse (2004)

"I cannot believe that anything so **ugly** *as multiplication of matrices is an essential part of the scheme of nature"*

Sir Arthur Eddington (1936)

— Thanks, Jeremy, for this quote :-)

Linear algebra in computer science

Trend towards **quantitative methods** in computer science (using **LA** in particular):

- Read Baroni and Zamparelli (2010) suggestive paper: *Nouns are vectors, adjectives are matrices* in semantics of natural language.
- "Quantum inspiration" in Sernadas et al. (2008) who regard probabilistic programs as linear transformations over suitable vector spaces.

Our own trend: PhD thesis by Hugo Macedo (2012) — just submitted — entitled

"Matrices as Arrows" — Why Categories of Matrices Matter"

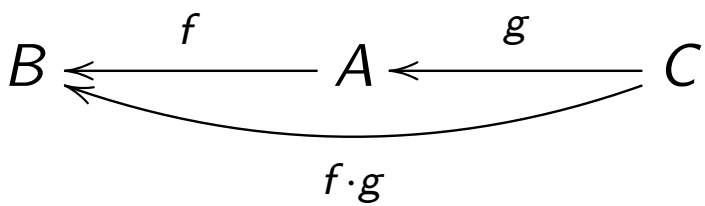
Arrow notation for functions

Used everywhere for declaring **functions**, eg.

$$\begin{array}{lcl} f & : & \mathbb{N} \to \mathbb{R} \\ n & \mapsto & \frac{n}{\pi} \end{array}$$

The first line is the **type** of the function (**syntax**) and the second line is the rule of correspondence (**semantics**).

Compositionality — functions compose with each other:



(7)

$$b = f(g\ c)$$

Arrow notation for (binary) relations

Binary relations are typed: arrow $A \xrightarrow{R} B$ denotes a binary relation from A (source) to B (target).

A, B are types. Writing $B \xleftarrow{R} A$ means the same as $A \xrightarrow{R} B$.

Compositionality — relations compose with each other:

$$B \xleftarrow{R} A \xleftarrow{S} C \qquad R \cdot S \tag{8}$$

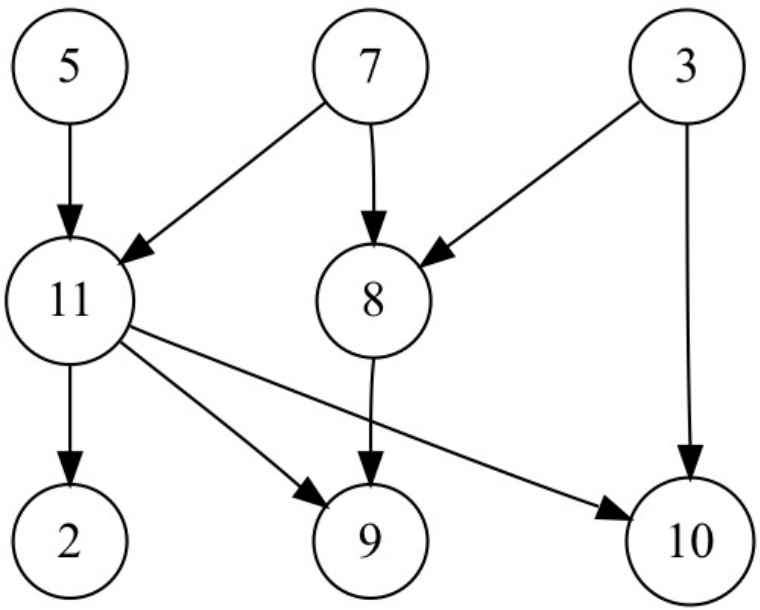
$$b(R \cdot S)c \;\Leftrightarrow\; \langle \exists\ a \;::\; b\ R\ a \wedge a\ S\ c \rangle \tag{9}$$

Example: $uncle = brother \cdot parent$

From binary relations to matrices

As binary relations are Boolean matrices, eg.

Relation R:



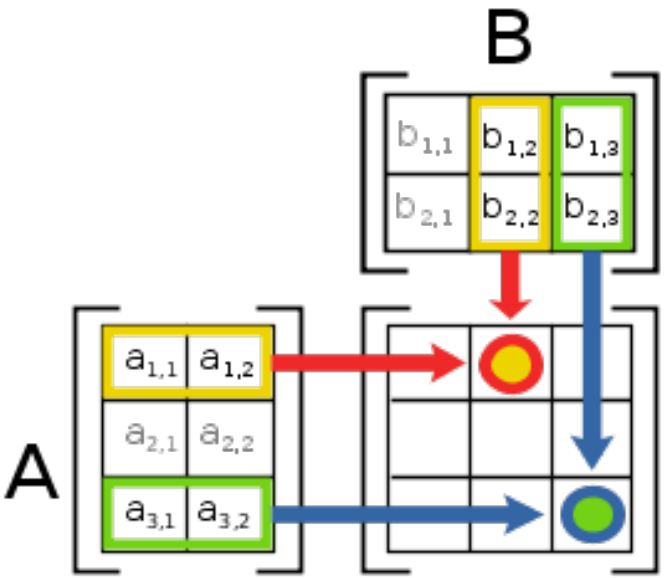
Matrix M:

	1	2	3	4	5	6	7	8	9	10	11
1	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	1
3	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0
8	0	0	1	0	0	0	1	0	0	0	0
9	0	0	0	0	0	0	0	1	0	0	1
10	0	0	1	0	0	0	0	0	0	0	1
11	0	0	0	0	1	0	1	0	0	0	0

why not represent matrices as arrows too, cf.

$$11 \xleftarrow{M} 11 \ ?$$

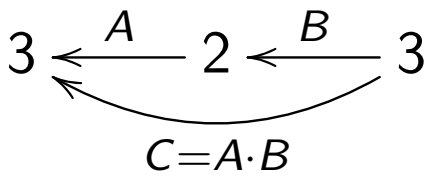
Compositionality — matrix-matrix multiplication



Index-wise definition

$$C_{ij} = \sum_{k=1}^{2} A_{ik} \times B_{kj}$$

Arrows hide indices i, j, k:

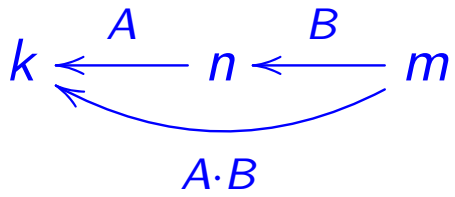


Index-free

$$C = A \cdot B$$

Typed linear algebra

Composition of matrices obeys to the **typing rule**



For matrices A and B **of the same type** $n \longleftarrow m$, we can extend cell level algebra to matrix level, eg. by **adding** and **subtracting** matrices,

$$A + B \quad , \quad A - B$$

multiplying matrices (the Hadamard product)

$$A \times B$$

and so on.

Typed linear algebra

Expressions such as eg. $A + B$, $A \times B$ for A and B of different types **won't typecheck**.

The underlying type system is **polymorphic** and type inference proceeds by **unification**, as in programming languages.

For instance, the **identity matrix**

$$n \xleftarrow{id_n} n \quad = \quad \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n \times n}$$

is polymorphic on type n. This could help in equipping tools such as MATLAB and MATHEMATICA with a type system saving the burden of always checking for matrix dimensions.

Converse

Given matrix $n \xleftarrow{M} m$, notation $m \xleftarrow{M^\circ} n$ denotes its transpose, or converse.

Operation whereby M changes shape by turning its rows into columns and vice-versa.

The following idempotence and contravariance laws hold:

$$(M^\circ)^\circ = M \tag{10}$$

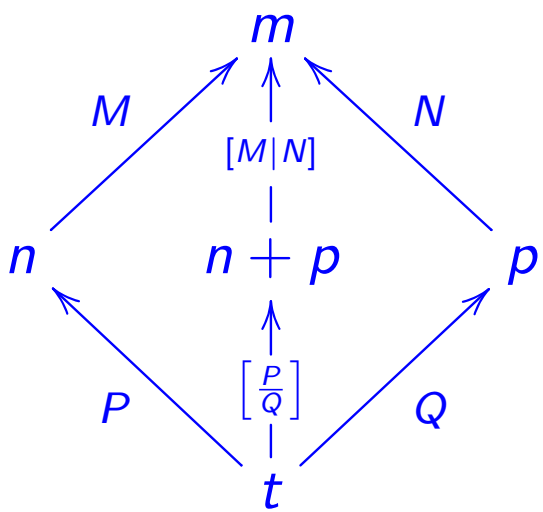
$$(M \cdot N)^\circ = N^\circ \cdot M^\circ \tag{11}$$

Polymorphic (block) combinators

Two ways of putting matrices together to build larger ones:

- $X = [M|N]$ — M and N side by side (“‘junc”)
- $X = \left[\frac{P}{Q}\right]$ — P on top of Q (“‘split”).

Mind the (polymorphic) types:



(A biproduct)

Blocked linear algebra

Rich set of laws, for instance

- **Divide-and-conquer:**

$$[A|B] \cdot \left[\frac{C}{D}\right] = A \cdot C + B \cdot D \tag{12}$$

- **"Fusion"-laws:**

$$C \cdot [A|B] = [C \cdot A|C \cdot B] \tag{13}$$

$$\left[\frac{A}{B}\right] \cdot C = \left[\frac{A \cdot C}{B \cdot C}\right] \tag{14}$$

Block-matrix-algebra is at the heart of **parallelism** in **LA** — this is why **LA**-based approaches are so important today.

Vectors

Vectors are special cases of matrices in which one of the types is 1, for instance

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \qquad \text{and} \qquad w = \begin{bmatrix} w_1 & \dots & w_n \end{bmatrix}$$

Column vector v is of type $m \longleftarrow 1$ (m rows, one column) and **row** vector w is of type $1 \longleftarrow n$ (one row, n columns).

Our convention is that lowercase letters (eg. v, w) denote vectors and uppercase letters (eg. A, M) denote arbitrary matrices.

Special matrices

The following (Boolean) matrices are relevant:

- The **bottom** matrix $n \xleftarrow{\bot} m$ — wholly filled with 0s
- The **top** matrix $n \xleftarrow{\top} m$ — wholly filled with 1s
- The **identity** matrix $n \xleftarrow{id} n$ — diagonal of 1s
- The **bang** (row) vector $1 \xleftarrow{!} m$ — wholly filled with 1s

Thus, (typewise) **bang** matrices are special cases of **top** matrices:

$$1 \xleftarrow{\top} m \quad = \quad !$$

Also note that, on type $1 \xleftarrow{} 1$:

$$\top \;=\; ! \;=\; id$$

Type generalization

As is standard is **relational mathematics** (Schmidt, 2010), matrix types can be generalized from numeric dimensions (n, $m \in \boldsymbol{N}_0$) to arbitrary denumerable types (X, Y), taking **disjoint union** $X + Y$ for $m + n$, Cartesian product $X \times Y$ for mn, etc.

We will restrict ourselves to matrices taking elements from $\boldsymbol{R}_0^+$, the non-negative reals.

The interval $[0, 1]$ will be particularly at target in the case of **probabilistic** functions.

Doing elementary set theory with LA

Clearly, any **Boolean** vector of type $1 \xleftarrow{!} X$ for some type X represents a subset of X, for instance

$$\begin{array}{c} \\ \mathbf{1} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{John} & \mathbf{Mary} & \mathbf{Henry} \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \end{array}$$

represents set $\{John, Henry\} \subseteq \{John, Mary, Henry\}$.

Let $[\![A]\!]$ denote the vector which represents $A \subseteq X$. Then:

$$\begin{aligned} [\![\emptyset]\!] = \bot \quad &, \quad [\![A]\!] = ! \\ [\![A \cap B]\!] &= [\![A]\!] \times [\![B]\!] \\ [\![\neg A]\!] &= ! - [\![A]\!] \\ [\![A \cup B]\!] &= [\![A]\!] + [\![B]\!] - [\![A \cap B]\!] \end{aligned} \tag{15}$$

Example of LA set theory calculation

State that A and B form a **partition** of X simply by writing

$$[\![A]\!] + [\![B]\!] = !$$

Infer that, necessarily, A and B are disjoint:

$$[\![A]\!] \times [\![B]\!] = \bot$$

For this, we only need to recall why **LA** is referred to as **linear**:

$$M \times (N + P) = M \times N + M \times P \quad (16)$$

$$M \cdot (N + P) = M \cdot N + M \cdot P \quad (17)$$

$$(N + P) \cdot M = N \cdot M + P \cdot M \quad (18)$$

etc, and drop the $[\![\_]\!]$ parentheses for saving ink:

Example of LA set theory calculation

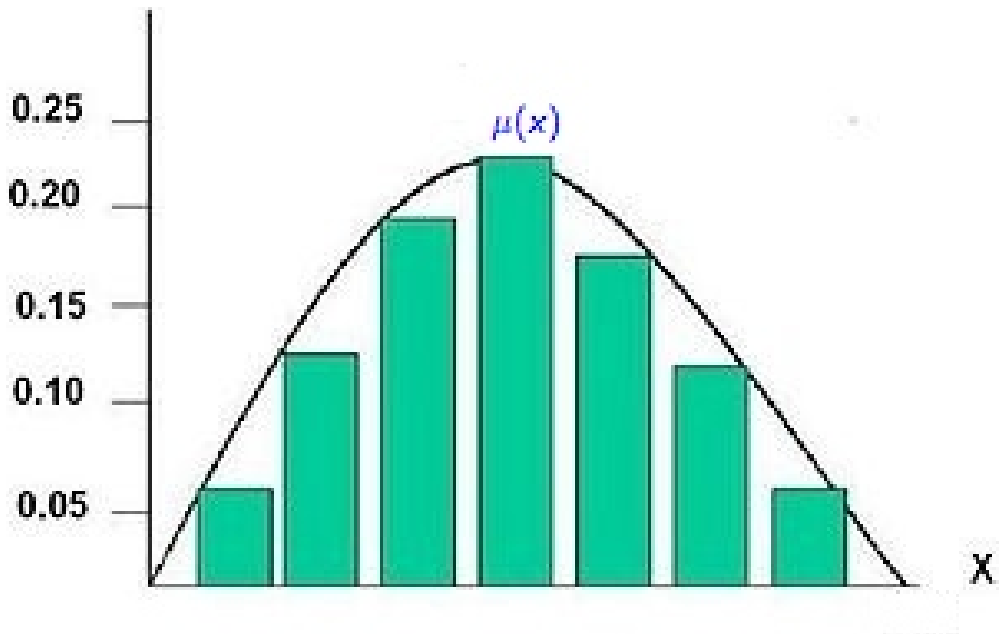
$$
\begin{array}{ll}
 & A + B = ! \\
\Rightarrow & \quad \{ \text{ multiply both sides by } A \text{ (Leibniz) } \} \\
 & A \times (A + B) = A \times ! \\
\Leftrightarrow & \quad \{ \text{ Hadamard: linearity (16), } A \times A = A \cap A = A \text{ and unit } ! \ \} \\
 & A + A \times B = A \\
\Leftrightarrow & \quad \{ \text{ subtract } A \text{ from both sides of the equation } \} \\
 & A \times B = A - A \\
\Leftrightarrow & \quad \{ \text{ cancellation of inverses } \} \\
 & A \times B = \bot \\
\Leftrightarrow & \quad \{ \text{ definition } \} \\
 & A \text{ and } B \text{ are disjoint}
\end{array}
$$

Doing elementary probability theory in LA

Recall standard definition of a **distribution**:

$$\mathcal{D}X \quad = \quad \{\mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1\} \tag{19}$$

as, for example,



Elementary probability theory in LA

The same in matrix format (for **discrete** μ) is column vector

$$X \xleftarrow{[\![\mu]\!]} 1$$

(over non-negative reals) such that $! \cdot [\![\mu]\!] = !$.

Why? The **sum** of all entries of vector $X \xleftarrow{v} 1$ is vector (cell)

$$1 \xleftarrow{!} X \xleftarrow{v} 1 \qquad !\cdot v$$

and $1 \xleftarrow{!} 1$ is real number 1.

Thus the pointfree style starts to happen (no need for $\sum$ etc)

Elementary probability theory in LA

Let $X \xrightarrow{\mu} [0,1]$ be a **discrete** probability **distribution** over sample space X.

The probability $P(S)$ of event $S \subseteq X$ under μ will be given by

$$P(S) = 1 \xleftarrow{[\![S]\!]} S \xleftarrow{[\![\mu]\!]} 1 \qquad (20)$$

$$[\![S]\!] \cdot [\![\mu]\!]$$

Then a *random variable* $T \xleftarrow{v} X$ on probability space (X, μ) induces a new probability distribution on T by composition,

$$[\![\mu']\!] = [\![v]\!] \cdot [\![\mu]\!] \qquad (21)$$

generating a new probability space (T, μ').

Elementary probability theory in LA

That μ' is indeed a distribution can be easily calculated:

$$
\begin{array}{ll}
 & ! \cdot [\![\mu']\!] \\
= & \quad \{ \ (21) \ \} \\
 & ! \cdot ([\![v]\!] \cdot [\![\mu]\!]) \\
= & \quad \{ \ v \text{ is a function, therefore Dirac-probabilistic} \ \} \\
 & ! \cdot [\![\mu]\!] \\
= & \quad \{ \ \mu \text{ is a distribution} \ \} \\
 & !
\end{array}
$$

Elementary probability theory in LA

Easy to get basic probability theory from the **LA** encoding, for instance the **addition law** of probability

$$P(A \cup B) \quad = \quad P(A) + P(B) - P(A \cap B)$$

coming straight from vectorial set union (15) and linearity (18).

Example: calculating the law of **total probability**

$$P(A) \quad = \quad P(A \cap B_1) + P(A \cap B_2)$$

where $\{B_1, B_2\}$ form a partition ($B_1 + B_2 = !$) of the sample space X (next slide).

Law of total probability

$$
\begin{aligned}
& P(A \cap B_1) + P(A \cap B_2) \\
= & \quad \{ \text{ (20) twice } \} \\
& (A \times B_1) \cdot \mu + (A \times B_2) \cdot \mu \\
= & \quad \{ \text{ composition and Hadamard product are bilinear } \} \\
& (A \times (B_1 + B_2)) \cdot \mu \\
= & \quad \{ \text{ partition : } B_1 + B_2 = ! \ \} \\
& (A \times !) \cdot \mu \\
= & \quad \{ \ ! = \top \text{ is the unit of Hadamard } \times \text{; definition (20) } \} \\
& P(A)
\end{aligned}
$$

Matrix transformed probabilistic functions

Recall that, given probabilistic function $A \xrightarrow{f} \mathcal{D}B$, its matrix **transform** is a matrix of type $A \xrightarrow{[\![f]\!]} B$ such that

$$! \cdot [\![f]\!] \quad = \quad ! \tag{22}$$

that is, all **columns** of $[\![f]\!]$ add up to one.

For $A = B$, probabilistic function f can be regarded as a **Markov chain**.

Example — probabilistic **negation**:

$$\begin{array}{c} \\ \textbf{True} \\ \textbf{False} \end{array} \begin{array}{c} \begin{array}{cc} \textbf{True} & \textbf{False} \end{array} \\ \begin{pmatrix} 0.1 & 0.8 \\ 0.9 & 0.2 \end{pmatrix} \end{array}$$

Linear algebra of probabilistic functions

Every sharp function is probabilistic — it offers the **Dirac distribution** for every input. This includes the identity function id represented by the identity matrix $[\![id]\!]$.

Compositionality: probabilistic functions compose, under monad-flavoured definition

$$[\![f \bullet g]\!] \quad = \quad [\![f]\!] \cdot [\![g]\!] \tag{23}$$

In monad-speak:

$$[\![\lambda \mathtt{a}.\ \mathbf{do}\ \{\mathtt{b} \leftarrow\ \mathtt{g}\ \mathtt{a}; \mathtt{f}\ \mathtt{b}\}\ ]\!] \quad = \quad [\![f]\!] \cdot [\![g]\!]$$

Let us not forget checking the 100% constraint (22).

$$
\begin{array}{ll}
 & f \bullet g \text{ is a probabilistic function} \\
\Leftrightarrow & \quad \{ \ (22) \ \} \\
 & ! \cdot [\![f \bullet g]\!] = ! \\
\Leftrightarrow & \quad \{ \ \text{definition (23)} \ \} \\
 & ! \cdot [\![f]\!] \cdot [\![g]\!] = ! \\
\Leftrightarrow & \quad \{ \ f \text{ is probabilistic (22)} \ \} \\
 & ! \cdot [\![g]\!] = ! \\
\Leftrightarrow & \quad \{ \ g \text{ is probabilistic (22)} \ \} \\
 & ! = !
\end{array}
$$

Probabilistic "junc"

Probabilistic $A + B \xrightarrow{[f,g]} \mathcal{D}C$ — run either f or g — transposes into

$$[\![[f,g]]\!] \quad = \quad [[\![f]\!]|[\![g]\!]] \tag{24}$$

where (recall) $[M|N]$ denotes M and N put **side by side**.

Checking the 100% constraint (22):

$$\begin{array}{ll} & !\cdot[[\![f]\!]|[\![g]\!]] \\ \Leftrightarrow & \quad \{ \text{ fusion-+ (13) } \} \\ & [!\cdot[\![f]\!]|!\cdot[\![g]\!]] \\ \Leftrightarrow & \quad \{ \text{ } f \text{ and } g \text{ probabilistic (22) } \} \\ & [!|!] \\ \Leftrightarrow & \quad \{ \text{ } [!|!] = ! \text{ } \} \\ & ! \end{array}$$

Probabilistic choice

In their programming language *pGCL*, McIver and Morgan (2005) introduce notation

$$prog \ {}_p\!\diamond \ prog'$$

as a form of **probabilistic choice** between two branches of a program, $prog$ chosen with probability p and $prog'$ with probability $1 - p$.

This corresponds to the choice between two probabilistic functions f and g **of the same type** defined by

$$[\![f \ {}_p\!\diamond \ g]\!] \quad = \quad p[\![f]\!] + (1-p)[\![g]\!] \tag{25}$$

Probabilistic choice

Probabilistic choice "is probabilistic":

$$
\begin{aligned}
& ! \cdot [\![f \ {}_p\diamond\ g]\!] \\
= & \quad \{ \text{ definition (25) ; bilinearity } \} \\
& ! \cdot (p[\![f]\!]) + ! \cdot ((1-p)[\![g]\!]) \\
= & \quad \{ \text{ } p \text{ is a scalar } \} \\
& p(! \cdot [\![f]\!]) + (1-p)(! \cdot [\![g]\!]) \\
= & \quad \{ \text{ } f \text{ and } g \text{ are probabilistic } \} \\
& p! + (1-p)! \\
= & \quad \{ \text{ bilinearity } \} \\
& (p + 1 - p)! \\
= & \quad \{ \text{ cancellation } \} \\
& !
\end{aligned}
$$

Properties

Probabilistic choice enjoys many properties easy to derive from the definition, eg. basic

$$f \;{}_p\!\diamond\; f = f \tag{26}$$
$$f \;{}_0\!\diamond\; g = g \tag{27}$$
$$f \;{}_p\!\diamond\; g = g \;{}_{1-p}\!\diamond\; f \tag{28}$$

fusion-laws

$$(f \;{}_p\!\diamond\; g) \bullet h = (f \bullet h) \;{}_p\!\diamond\; (g \bullet h) \tag{29}$$
$$h \bullet (f \;{}_p\!\diamond\; g) = (h \bullet f) \;{}_p\!\diamond\; (h \bullet g) \tag{30}$$

and the **exchange** law:

$$[f, g] \;{}_p\!\diamond\; [h, k] = [f \;{}_p\!\diamond\; h, g \;{}_p\!\diamond\; k] \tag{31}$$

Probabilistic sums

The **direct sum** of two matrices,

$$M \oplus N = [i_1 \cdot M | i_2 \cdot N] = \left[\frac{M \cdot \pi_1}{N \cdot \pi_2}\right] = \left[\begin{array}{c|c} M & 0 \\ \hline 0 & N \end{array}\right] \quad (32)$$

which has type $\begin{array}{ccc} A & B & A+B \\ {\scriptstyle M}\downarrow & {\scriptstyle N}\downarrow & \downarrow{\scriptstyle M \oplus N} \\ C & D & C+D \end{array}$ (a **bifunctor**) enables us to sum probabilistic functions:

$$[\![f \oplus g]\!] = [\![f]\!] \oplus [\![g]\!]$$

Distribution over choice

$$h \oplus (f \; {}_p\diamond \; g) = (h \oplus f) \; {}_p\diamond \; (h \oplus g) \quad (33)$$

is central to probabilistic function calculation.

Probabilistic recursion

Recall the algorithm for performing **multiplication** by **iterated addition** (over $\mathbb{N}_0$),

$$(a*) = \textbf{for}\ (a+)\ 0$$

which we wrote in two clauses

$$a * 0 = 0$$
$$a * (n + 1) = a + a * n$$

earlier on, only to inject a fault into the base case:

$$a * 0 =_{.95} 0$$
$$a * 0 =_{.05} a$$
$$a * (n + 1) =_{1} a + a * n$$

Probabilistic recursion

Now that we have probabilistic choice (${}_p\diamond$) we can encode the faulty version as follows (also parametric on the probability p of the fault),

$$faulty = \textbf{for}\ (a+)\ (0\ {}_p\diamond\ a)$$

and **try and quantify** how bad it is if compared to the original version,

$$good = \textbf{for}\ (a+)\ 0$$

or to one definitely bad:

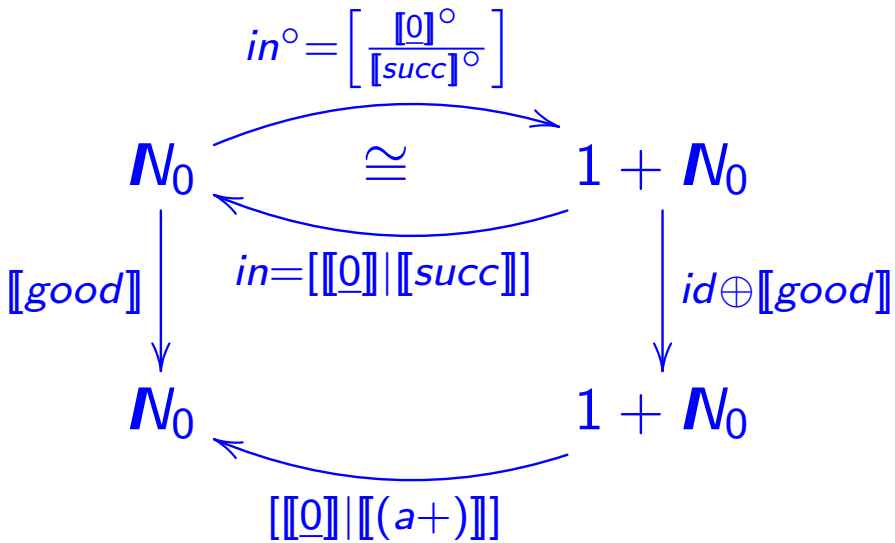
$$bad = \textbf{for}\ (a+)\ a$$

(Matrix) catamorphisms

Note the slight abuse of notation, as we don't have a choice for values — only for functions. It should be:

$$faulty = (\!|[\underline{0}\ {}_p\diamond\ \underline{a}|(a+)]|\!)$$
$$bad = (\!|[\underline{a}|(a+)]|\!)$$
$$good = (\!|[\underline{0}|(a+)]|\!)$$

Here is how *good* looks like in a diagram:



(Matrix) catamorphisms

Parsing the diagram, one has:

$$[\![good]\!] = [[\![\underline{0}]\!]|[\![(a+)]\!]] \cdot (id \oplus [\![good]\!] \cdot \left[\frac{[\![\underline{0}]\!]^\circ}{[\![succ]\!]^\circ}\right]$$

$$= \quad \{ \text{ absorption (34) ; dropping parentheses for better parsing } \}$$

$$good = [\underline{0}|(a+) \cdot good] \cdot \left[\frac{\underline{0}^\circ}{succ^\circ}\right]$$

$$= \quad \{ \text{ divide and conquer (12) } \}$$

$$good = \underline{0} \cdot \underline{0}^\circ + (a+) \cdot good \cdot succ^\circ$$

Note how the matrix for *good* is recursively filled up: first the outer-product $\underline{0} \cdot \underline{0}^\circ$ (that is, the everywhere-0 matrix apart from the 1 in cell $(0,0)$), which is added to $(a+) \cdot \underline{0} \cdot \underline{0}^\circ \cdot succ^\circ = \underline{a} \cdot \underline{1}^\circ$ (matrix with a 1 in cell labeled $(a,1)$ and 0 otherwise), and so on.

Checking assertion

Conjecture: is it true that

$$faulty = good \ {}_p\diamond \ bad$$

We reason:

$$\begin{array}{ll} & faulty = good \ {}_p\diamond \ bad \\ \Leftrightarrow & \quad \{ \text{ definition of } faulty \ \} \\ & (\!|[\underline{0} \ {}_p\diamond \ \underline{a}|(a+)]|\!) = good \ {}_p\diamond \ bad \\ \Leftrightarrow & \quad \{ \text{ cata-universal property, for } \mathsf{F}X = id \oplus X \ \} \\ & (good \ {}_p\diamond \ bad) \cdot [\underline{0}|succ] = [\underline{0} \ {}_p\diamond \ \underline{a}|(a+)] \cdot \mathsf{F}(good \ {}_p\diamond \ bad) \end{array}$$

The calculation of this equality will resort to $\oplus$-absorption:

$$[M|N] \cdot (P \oplus Q) \quad = \quad [M \cdot P|N \cdot Q] \qquad (34)$$

Checking assertion

$$
\begin{aligned}
& (good \ {}_p\diamond \ bad) \cdot [\underline{0}|succ] \\
= & \quad \{ \ {}_p\diamond\text{-fusion (29)} \ \} \\
& (good \cdot [\underline{0}|succ]) \ {}_p\diamond \ (bad \cdot [\underline{0}|succ]) \\
= & \quad \{ \ \text{cata-cancellation (twice: } good \text{ and } bad) \ \} \\
& ([\underline{0}|(a+)] \cdot \mathsf{F}good) \ {}_p\diamond \ ([\underline{a}|(a+)] \cdot \mathsf{F}bad) \\
= & \quad \{ \ \text{absorption (34) over } \mathsf{F}X = id \oplus X \ \} \\
& [\underline{0}|(a+) \cdot good] \ {}_p\diamond \ [\underline{a}|(a+) \cdot bad] \\
= & \quad \{ \ \text{exchange-law: } [f,g] \ {}_p\diamond \ [h,k] = [f \ {}_p\diamond \ h, g \ {}_p\diamond \ k] \ (31) \ \} \\
& [\underline{0} \ {}_p\diamond \ \underline{a} \mid ((a+) \cdot good) \ {}_p\diamond \ ((a+) \cdot bad)]
\end{aligned}
$$

Checking assertion

$$
\begin{array}{ll}
= & \{\ \text{fusion (30)}\ \} \\
& [\underline{0}\ {}_p\diamond\ \underline{a} \mid (a+)\cdot(good\ {}_p\diamond\ bad)] \\
= & \{\ \oplus\text{-absorption (34), in the reverse direction}\ \} \\
& [\underline{0}\ {}_p\diamond\ \underline{a}|(a+)]\cdot \mathsf{F}(good\ {}_p\diamond\ bad) \\
\square &
\end{array}
$$

Note how exchange-law (31)

$$[f, g]\ {}_p\diamond\ [h, k] = [f\ {}_p\diamond\ h, g\ {}_p\diamond\ k]$$

plays the central role above, on top of bilinearity.

Closing

The research question which motivated this talk splits in two other questions, in fact two sides of the same coin:

(a) Can **the** AoP be extended quantitatively in some useful way?
(b) What happens to the discipline once we generalize from relations to matrices?

The answer leads us into **linear algebra**, which eventually provides a surprisingly simple framework for calculating with **set-theory**, **probabilities**, functions and relations, provided it is **typed** — as advocated by Macedo (2012).

Closing

The comment by Sir Arthur Eddington in his *Relativity Theory of Electrons and Protons* (already quoted) can be understood as a call for better laid out **linear algebra** — perhaps **typed** :-)?

Is this kind of foundation sought in 1967, in the Garmisch NATO workshop:

> *In late 1967 the Study Group recommended the holding of a working conference on Software Engineering. The phrase 'software engineering' was deliberately chosen as being* **provocative**, *in implying the need for software manufacture to be based on the types of* **theoretical foundations** *and practical disciplines, that are traditional in the established branches of engineering. (Naur and Randell, 1969)*

Only **time** and **experience** will tell.

Future work

Topic in its very start (very recent research, a couple of months), much to be done:

- **LAoP** needs to be pushed forward to realistic case-studies.
- Typed LA framework applied to **OLAP** and data-mining — follow up of work already carried out by (Macedo and Oliveira, 2011). Links to **parallel programming**
- Extension of the **GNU Octave** (`http://www.gnu.org/software/octave`) matrix sub-language with **types** and a **type checker** along the lines of this talk.

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