Completeness and Incompleteness in nominal Kleene algebra

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Context

- Names are pervasive in computer science;
- Semantics of programming languages (α-equivalence);

$$f(a) = 2 * a \quad g(b) = 2 * b$$

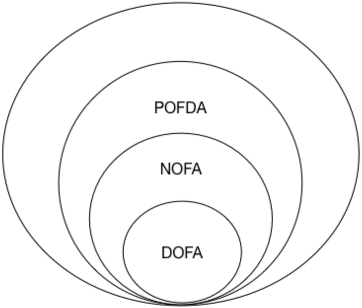
- Range of proposals for sound semantics: Pistore-Montanari, Gabbay-Pitts, ...
- Nominal sets (Fraenkel and Mostowski, early twentieth century).

Context

- Francez and Kaminski: finite memory automata.
- Montanari and Pistore: HD-automata.
- Murawski and Tzevelekos: fresh-register automata.
- Bojanczyk, Klin, Lasota: extensive results on nominal automata theory.
- Gabbay and Ciancia: nominal Kleene algebras.
- Kurz, Suzuki, Tuosto: regular expressions for HD-automata.

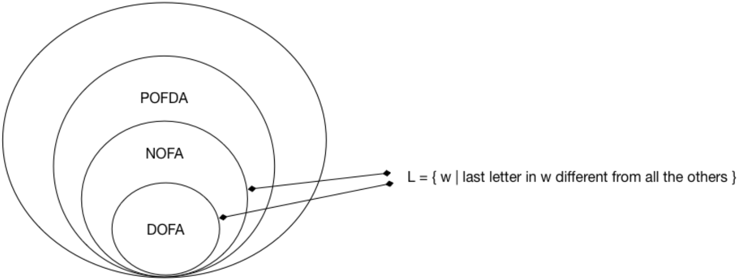
Nominal Chomsky hierarchy

Key point in Polish work: new notion of finiteness, orbit-finiteness.

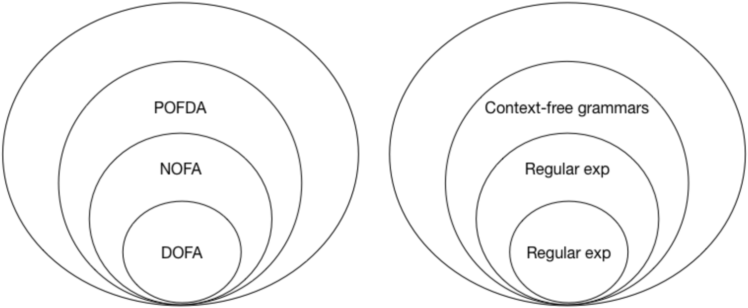


Nominal Chomsky hierarchy

Unexpected things happen with orbit-finiteness.

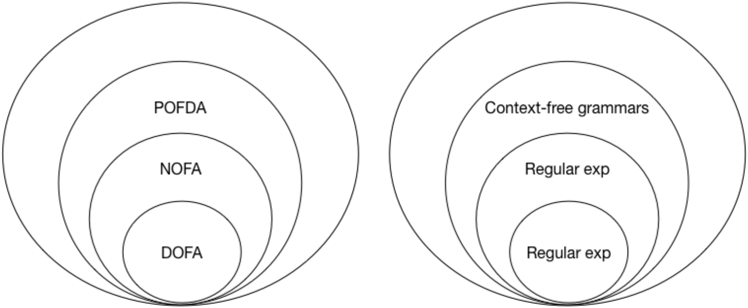


Nominal Chomsky hierarchy



Language hierarchy and correspondence theorems?

Nominal Chomsky hierarchy



Murawski (June 2015): to this day we still do not have a satisfactory notion of nominal regular language.

This talk: some results, many problems. . .

- Nominal Kleene algebra (Ciancia & Gabbay) axioms are not complete.
- Characterisation of the free Nominal Kleene algebra.

- Nominal Kleene algebra does not give a Kleene Theorem.
- New automaton model for one-sided Kleene Theorem.

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Nominal Sets [Gabbay & Pitts, LICS 1999]

Nominal Sets

- a convenient framework for name generation, binding, α-conversion

Applications

- logic: quantifiers
- programming language semantics: references, objects, pointers, function parameters
- XML document processing
- cryptography: nonces

Group Action

- Let G be a group and X a set
- A group action of G on X is a map $G \times X \to X$ such that

$$\pi(\rho x) = (\pi\rho)x \qquad\qquad 1x = x$$

- A G-set is a set X equipped with a group action $G \times X \to X$
- $f : X \to Y$ is equivariant if $f \circ \pi = \pi \circ f$ for all $\pi \in G$

Nominal Sets

- Let $\mathbb{A}$ be a countably infinite set of atoms
- Let G be the group of all finite permutations of $\mathbb{A}$ (permutations generated by transpositions (ab))
- If G acts on X, say that $A \subseteq \mathbb{A}$ supports $x \in X$ if

$$\mathrm{Fix}\, A \subseteq \mathrm{fix}\, x$$

where $\mathrm{fix}\, x = \{\pi \in G \mid \pi x = x\}$ and $\mathrm{Fix}\, A = \bigcap_{x \in A} \mathrm{fix}\, x$

Nominal Sets

- $x \in X$ has finite support if there is a finite $A \subseteq \mathbb{A}$ that supports x
- If $x \in X$ has finite support, then it has a minimum supporting set $\operatorname{supp} x$, the support of x
- Write $a \# x$ and say a is fresh for x if $a \notin \operatorname{supp} x$
- A nominal set is a set X with a group action of G such that every element has finite support

Nominal Sets

Example

- $\mathbb{A} = \{\text{variables}\}$
- $X = \{\lambda\text{-terms over } \mathbb{A}\}$
- If $\pi \in G$ and $\pi a = a$ for $a \in \mathrm{FV}(x)$, then $\pi x = x$ (α-conversion)
- $A \subseteq \mathbb{A}$ supports $x \iff \mathrm{FV}(x) \subseteq A$
- $\operatorname{supp} x = \mathrm{FV}(x)$
- $a \# x$ iff $a \notin \mathrm{FV}(x)$

$$(b\ c)\ ((\lambda b.a(bb))(\lambda b.a(bb))) = (\lambda c.a(cc))(\lambda c.a(cc))$$

More examples

- The set $\mathbb{A}$ is a G-set under the group action $\pi a = \pi(a)$. It is a nominal set with $\mathrm{supp}(a) = \{a\}$.
- The set $\mathcal{P}\mathbb{A}$ is a G-set, but not a nominal set.
- The set $\mathcal{P}_{fs}\mathbb{A}$ of finite and co-finite subsets of $\mathbb{A}$ is a nominal set.

Kleene Algebra

Idempotent Semiring Axioms

$$p + (q + r) = (p + q) + r \qquad p(qr) = (pq)r$$
$$p + q = q + p \qquad 1p = p1 = p$$
$$p + 0 = p \qquad p0 = 0p = 0$$
$$p + p = p$$
$$p(q + r) = pq + pr \qquad a \leq b \stackrel{\triangle}{\Longleftrightarrow} a + b = b$$
$$(p + q)r = pr + qr$$

Axioms for *

$$1 + pp^* \leq p^* \qquad q + px \leq x \;\Rightarrow\; p^*q \leq x$$
$$1 + p^*p \leq p^* \qquad q + xp \leq x \;\Rightarrow\; qp^* \leq x$$

Standard Model

Regular sets of strings over Σ

$$
\begin{array}{rcl}
A + B & = & A \cup B \\
AB & = & \{xy \mid x \in A,\ y \in B\} \\
A^* & = & \bigcup_{n \geq 0} A^n \quad = \quad A^0 \cup A^1 \cup A^2 \cup \cdots \\
1 & = & \{\varepsilon\} \\
0 & = & \emptyset
\end{array}
$$

This is the free KA on generators Σ

Other Models

- Relational models
- Trace models used in semantics
- (min, +) algebra used in shortest path algorithms
- (max, +) algebra used in coding
- Convex sets used in computational geometry (Iwano & Steiglitz 90)
- Matrix algebras

Nominal KA [Gabbay & Ciancia 2011]

A nominal Kleene algebra (NKA) over atoms $\mathbb{A}$ is a structure

$$(K, +, \cdot, ^*, 0, 1, \nu)$$

with $\nu : \mathbb{A} \times K \to K$ such that

- K is a nominal set over $\mathbb{A}$
- the KA operations and ν are equivariant:

$$\begin{aligned} \pi(x+y) &= \pi x + \pi y & \pi(0) &= 0 \\ \pi(xy) &= (\pi x)(\pi y) & \pi(1) &= 1 \\ \pi(x^*) &= (\pi x)^* & \pi(\nu a.e) &= \nu(\pi a).\pi e \end{aligned}$$

  equivalently, every $\pi \in G$ is an automorphism of K
- all the KA axioms are satisfied and ν satisfies...

Nominal Axioms [Gabbay & Ciancia 2011]

Odersky style axioms	interaction with KA operators
$a\#e \Rightarrow \nu a.e = e$	$\nu a.(d+e) = \nu a.d + \nu a.e$
$\nu a.\nu b.e = \nu b.\nu a.e$	$a\#e \Rightarrow (\nu a.d)e = \nu a.de$
$a\#e \Rightarrow \nu b.e = \nu a.(a\ b)e$	$a\#e \Rightarrow e(\nu a.d) = \nu a.ed$

Nominal KA [Gabbay & Ciancia 2011]

Expressions

$$e ::= a \in \Sigma \mid e + e \mid ee \mid e^* \mid 0 \mid 1 \mid \nu a.e$$

The operator νa is a binding operator whose scope is e

The set of expressions over Σ is denoted Exp_Σ

ν-strings

A ν-string is an expression with no occurrence of $+$, *, 0, or 1 (except to denote the null string, in which case we use ε)

$$x ::= a \in \Sigma \mid xx \mid \varepsilon \mid \nu a.x$$

The set of ν-strings over Σ is denoted Σ^ν.

Nominal Language Model [Gabbay & Ciancia 2011]

$$NL : \mathsf{Exp}_{\mathbb{A}} \to \mathcal{P}(\mathbb{A}^*)$$

Example:

$$NL(\nu a.ab) = \{ab \mid a \neq b\}$$

$$NL((\nu a.ab)(\nu a.ab)) = \{abcb \mid a, c \in \mathbb{A} \text{ distinct and different than } b\}$$

Care must be taken when defining product to avoid capture!

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Intermediate interpretation as sets of ν-strings over $\mathbb{A}$

$$I : \mathrm{Exp}_{\mathbb{A}} \to \mathcal{P}(\mathbb{A}^{\nu})$$

$+$, $\cdot$, *, 0, and 1 have their usual set-theoretic interpretations, and

$$I(\nu a.e) = \{\nu a.x \mid x \in I(e)\} \qquad\qquad I(a) = \{a\}.$$

Examples

$$I(\nu a.a) = \{\nu a.a\}$$
$$I(\nu a.\nu b.(a+b)) = \{\nu a.\nu b.a, \nu a.\nu b.b\}$$
$$I(\nu a.(\nu b.ab)(a+b)) = \{\nu a.(\nu b.ab)a, \nu a.(\nu b.ab)b\}$$
$$I(\nu a.(ab)^*) = \{\nu a.\varepsilon, \nu a.ab, \nu a.abab, \nu a.ababab, \ldots\}$$
$$I((\nu a.ab)^*) = \{\varepsilon, \nu a.ab, (\nu a.ab)(\nu a.ab), (\nu a.ab)(\nu a.ab)(\nu a.a$$

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\end{aligned}$$

Nominal Language Model [Gabbay & Ciancia 2011]

$$NL : \mathbb{A}^\nu \to \mathcal{P}(\mathbb{A}^*)$$

- α-convert so that all bindings in x are distinct and different from free variables in x
- delete all binding operators νa to obtain $x' \in \mathbb{A}^*$
- $NL(x) = \{\pi(x') \mid \pi \in \mathrm{fix}\,\mathrm{FV}(x)\}$
- $NL(e) = \bigcup_{x \in I(e)} NL(x)$

Example

$$NL((\nu a.ab)(\nu a.ab)(\nu a.ab))$$
$$= \{abcbdb \mid a, c, d \in \mathbb{A} \text{ distinct and different from } b\}$$

Completeness and Incompleteness

Lemma

For $x, y \in \mathbb{A}^{\nu}$, $\vdash x = y$ implies $NL(x) = NL(y)$.

Incompleteness

$\nvdash a \leq \nu a.a$ but $NL(a) = \{a\} \subseteq \mathbb{A} = NL(\nu a.a)$

Alternative Nominal Language Model

Let Σ and $\mathbb{A}$ be countably infinite disjoint sets, $a, b, c, \ldots \in \mathbb{A}$, $x, y, z, \ldots \in \Sigma$, and $u, v, w, \ldots \in (\Sigma \cup \mathbb{A})^*$. Quantification is only over Σ.

A language is a subset $A \subseteq (\Sigma \cup \mathbb{A})^*$ such that $\pi A = A$ for all $\pi \in G$. The set of languages is denoted $\mathcal{L}$.

$$AB = \{uv \mid u \in A,\ v \in B,\ \mathrm{FV}(u) \cap \mathrm{FV}(v) \cap \mathbb{A} = \emptyset\}$$
$$\nu x.A = \{w[a/x] \mid w \in A,\ a \in \mathbb{A} - \mathrm{FV}(w)\},\ x \in \Sigma$$

Completeness

Theorem

The axioms of nominal Kleene algebra are sound and complete for the equational theory of nominal Kleene algebras and for the equational theory of the alternative language model:

$$\vdash e_1 = e_2 \iff AL(e_1) = AL(e_2)$$

The alternative language model is the free nominal KA.

Completeness

- exposing bound variables
- scope configuration
- canonical choice of bound variables
- semilattice identities

Determining Semilattice Identities

- Any substring of the form $\nu a.x$ of a ν-string generated by e_1 or e_2 must be generated by a subexpression $\nu a.d$
- There may be several different subexpressions of this form
- The sets of ν-strings generated by the ν-subexpressions could satisfy various semilattice identities, and we may have to know these identities to prove equivalence

Determining Semilattice Identities

Example

Consider $c_1 + c_2$ and $d_1 + d_2 + d_3$, where

$$c_1 = \nu a.a(aa)^* \qquad d_1 = \nu a.a(aaa)^*$$
$$c_2 = \nu a.aa(aa)^* \qquad d_2 = \nu a.aa(aaa)^*$$
$$d_3 = \nu a.aaa(aaa)^*$$

- c_i generates strings with i mod 2 a's
- d_i generates strings with i mod 3 a's
- Both $c_1 + c_2$ and $d_1 + d_2 + d_3$ generate all nonempty strings of a's, but in different ways

Determining Semilattice Identities

Express every ν-subexpression in e_1 or e_2 as a sum of atoms of the Boolean algebra on sets of ν-strings generated by these ν-subexpressions.

In the example above, the atoms are

$$b_i = \nu a.a^i(a^6)^*,\ 1 \le i \le 6$$

so b_i generates strings with i mod 6 a's.

Determining Semilattice Identities

Rewriting as sums of atoms,

$$
\begin{aligned}
c_1 &= b_1 + b_3 + b_5 & \qquad d_1 &= b_1 + b_4 \\
c_2 &= b_2 + b_4 + b_6 & \qquad d_2 &= b_2 + b_5 \\
& & \qquad d_3 &= b_3 + b_6 .
\end{aligned}
$$

The equivalences are provable in pure KA plus the nominal axiom $\nu a.(d + e) = \nu a.d + \nu a.e$. This gives

$$
\begin{aligned}
c_1 + c_2 &= (b_1 + b_3 + b_5) + (b_2 + b_4 + b_6) \\
d_1 + d_2 + d_3 &= (b_1 + b_4) + (b_2 + b_5) + (b_3 + b_6)
\end{aligned}
$$

Determining Semilattice Identities

Now observe

- any ν-string $\nu a.x$ generated by e_1 or e_2 is generated by exactly one atom $\nu a.e$
- we can treat $\nu a.e$ as atomic!
- we can even replace each atom $\nu a.e$ by a single letter $a_{\nu a.e}$ in e_1 and e_2, and the resulting expressions are equivalent, therefore provable
- for expressions of ν-depth greater than one, perform the construction inductively, innermost scopes first

Conclusions

- free language model consisting of regular sets of ν-strings modulo the Gabbay–Ciancia axioms
- new techniques in the completeness proof, e.g.
  - The Boolean algebra generated by finitely many regular sets consists of regular sets and is atomic.
  - Crucial for the normal form: every expression can be written as a sum of atoms.

Conclusions

- nominal versions of the syntactic and semantic Brzozowski derivative
- finitely supported sets of ν-strings modulo the Gabbay–Ciancia axioms form the final coalgebra
- half a Kleene theorem (expressions $\Rightarrow$ automata)
- exponential space decision procedure

Open Problems

- Complexity?
- Other half of the Kleene theorem is false:

$$\begin{tikzcd}
 & s_1(a,b) \arrow[dr, "a"] & \\
s_0(a) \arrow[ur, "\nu b"] & & s_0(b) \arrow[dl, "\nu a"] \\
 & s_1(b,a) \arrow[ul, "b"] &
\end{tikzcd}$$

The set of ν-strings accepted from state $s_0(a)$ is

$$\{\varepsilon,\ \nu b.ba,\ \nu b.ba(\nu a.ab),\ \nu b.ba(\nu a.ab(\nu b.ba)),\ \nu b.ba(\nu a.ab(\nu b.ba(\nu a.ab))), \ldots\}$$

Requires unbounded ν-depth!

Open Problems

- Can we characterize bounded ν-depth automata in a way that would lead to a converse of the Kleene theorem?
- Can we extend the syntax of expressions to capture sets of unbounded ν-depth? Yes:

$$X_a = \varepsilon + \nu b.bY_{ab} \qquad\qquad Y_{ab} = aX_b$$

...but this leaves us with the task of providing proof rules and proving completeness