

MFES/CSI — RELATION CALCULUS REFERENCE SHEET

RELATIONAL COMPOSITION

$$\text{Pointwise def.} \quad \begin{array}{c} B \xleftarrow{R} A \xleftarrow{S} C \\ \quad \quad \quad \curvearrowleft \\ \quad \quad \quad R \cdot S \end{array} \quad b(R \cdot S)c \equiv \langle \exists a : b R a : a S c \rangle \quad (5.11)$$

$$\text{Associativity} \quad R \cdot (S \cdot P) = (R \cdot S) \cdot P \quad (5.12)$$

$$\text{Identity} \quad \begin{cases} R = R \cdot id_A \\ R = id_B \cdot R \end{cases} \quad (5.13)$$

CONVERSE

$$\text{Pointwise def.} \quad b R a \Leftrightarrow a R^\circ b \quad (5.14)$$

$$\text{Universal-}^\circ \quad X^\circ \subseteq Y \equiv X \subseteq Y^\circ \quad (5.138)$$

$$\text{Involution} \quad (R^\circ)^\circ = R \quad (5.15)$$

$$\text{Contravariance} \quad (R \cdot S)^\circ = S^\circ \cdot R^\circ \quad (5.16)$$

$$\text{Isomorphism} \quad R \subseteq S \equiv R^\circ \subseteq S^\circ \quad (5.139)$$

$$\text{"Guardanapo"} \quad b(f^\circ \cdot R \cdot g)a \equiv (f b)R(g a) \quad (5.17)$$

RELATION INCLUSION

$$\text{Pointwise} \quad R \subseteq S \text{ iff } \langle \forall a, b :: b R a \Rightarrow b S a \rangle \quad (5.19)$$

$$\text{Reflexion} \quad R \subseteq R \quad (5.21)$$

$$\text{Transitivity} \quad R \subseteq S \wedge S \subseteq T \Rightarrow R \subseteq T \quad (5.22)$$

$$\text{Top and bottom} \quad \perp \subseteq R \subseteq \top \quad (5.25)$$

$$\text{Absorption} \quad R \cdot \perp = \perp \cdot R = \perp \quad (5.26)$$

RELATION EQUALITY

$$\text{Pointwise} \quad R = S \text{ iff } \langle \forall a, b : a \in A \wedge b \in B : b R a \Leftrightarrow b S a \rangle \quad (5.18)$$

$$\text{Indirect equality} \quad \begin{aligned} R = S &\equiv \langle \forall X :: (X \subseteq R \Leftrightarrow X \subseteq S) \rangle \\ &\equiv \langle \forall X :: (R \subseteq X \Leftrightarrow S \subseteq X) \rangle \end{aligned} \quad (5.24)$$

$$\text{"Ping-pong"} \quad R = S \equiv R \subseteq S \wedge S \subseteq R \quad (5.20)$$

RELATION TAXONOMY

Kernel $\ker R = R^\circ \cdot R$ (5.32)

Image $\text{img } R \stackrel{\text{def}}{=} R \cdot R^\circ$ (5.33)

Duality $\ker (R^\circ) = \text{img } R$
 $\text{img } (R^\circ) = \ker R$ (5.34, 5.35)

Criteria

	<i>Reflexive</i>	<i>Coreflexive</i>
$\ker R$	entire R	injective R
$\text{img } R$	surjective R	simple R

(5.36)

FUNCTIONS

Shunting rules $f \cdot R \subseteq S \equiv R \subseteq f^\circ \cdot S$
 $R \cdot f^\circ \subseteq S \equiv R \subseteq S \cdot f$ (5.46, 5.47)

Equality $f \subseteq g \equiv f = g \equiv f \supseteq g$ (5.48)

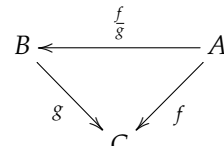
CONSTANT FUNCTIONS

Natural property $\underline{k} \cdot R \subseteq \underline{k}$ (5.39)

Corollary $\ker ! = \frac{!}{!} = \top$ (5.—)

Truth functions $\text{true} = \underline{\text{True}}$
 $\text{false} = \underline{\text{False}}$ (5.40, 5.41)

FUNCTION DIVISION

Definition $\frac{f}{g} = g^\circ \cdot f$ cf.  (5.49)

Properties

$$\begin{aligned} \frac{f}{\text{id}} &= f \\ \left(\frac{f}{g}\right) &= \frac{g}{f} \\ \frac{f \cdot h}{g \cdot k} &= k^\circ \cdot \frac{f}{g} \cdot h \\ \frac{f}{f} &= \ker f \\ a \neq b &\Leftrightarrow \frac{a}{b} = \perp \end{aligned}$$

(5.50 → 5.54)

RELATION UNION

Pointwise definition $b (R \cup S) a \equiv b R a \vee b S a$ (5.57)

Universal property $R \cup S \subseteq X \equiv R \subseteq X \wedge S \subseteq X$ (5.59)

Right linearity $R \cdot (S \cup T) = (R \cdot S) \cup (R \cdot T)$ (5.60)

$$\text{Left linearity} \quad (S \cup T) \cdot R = (S \cdot R) \cup (T \cdot R) \quad (5.61)$$

$$\text{Converse-}\cup \quad (R \cup S)^\circ = R^\circ \cup S^\circ \quad (5.65)$$

$$\text{Top-}\cup \quad R \cup T = T \quad (5.68)$$

$$\text{Bottom-}\cup \quad R \cup \perp = R \quad (5.69)$$

$$\text{Union simplicity} \quad M \cup N \text{ is simple} \equiv M, N \text{ are simple and } M \cdot N^\circ \subseteq id \quad (5.70)$$

RELATION INTERSECTION

$$\text{Pointwise definition} \quad b(R \cap S)a \equiv bRa \wedge bSa \quad (5.56)$$

$$\text{Universal property} \quad X \subseteq R \cap S \equiv X \subseteq R \wedge X \subseteq S \quad (5.58)$$

$$\text{Distribution (1)} \quad (S \cap Q) \cdot R = (S \cdot R) \cap (Q \cdot R) \Leftrightarrow \begin{cases} Q \cdot \text{img } R \subseteq Q \\ \vee \\ S \cdot \text{img } R \subseteq S \end{cases} \quad (5.62)$$

$$\text{Distribution (2)} \quad R \cdot (Q \cap S) = (R \cdot Q) \cap (R \cdot S) \Leftrightarrow \begin{cases} (\ker R) \cdot Q \subseteq Q \\ \vee \\ (\ker R) \cdot S \subseteq S \end{cases} \quad (5.63)$$

$$\text{Distribution (3)} \quad g^\circ \cdot (R \cap S) \cdot f = g^\circ \cdot R \cdot f \cap g^\circ \cdot S \cdot f \quad (5.71)$$

$$\text{Converse-}\cap \quad (R \cap S)^\circ = R^\circ \cap S^\circ \quad (5.64)$$

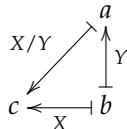
$$\text{Bottom-}\cap \quad R \cap \perp = \perp \quad (5.66)$$

$$\text{Top-}\cap \quad R \cap T = R \quad (5.67)$$

$$\text{Misc.} \quad k^\circ \cdot (f \cup g) = \frac{f}{k} \cup \frac{g}{k}, \quad k^\circ \cdot (f \cap g) = \frac{f}{k} \cap \frac{g}{k} \quad (5.73)$$

RELATION DIVISION

$$\text{Universal-}/ \quad Z \cdot Y \subseteq X \equiv Z \subseteq X/Y \quad (5.159)$$

$$\text{Pointwise-}/ \quad c(X/Y)a \equiv \langle \forall b : a Y b : c X b \rangle$$


$$(5.160)$$

$$\text{Universal-}\backslash \quad X \cdot Z \subseteq Y \Leftrightarrow Z \subseteq X \backslash Y \quad (5.161)$$

$$\text{Pointwise-}\backslash \quad a(X \backslash Y)c \equiv \langle \forall b : b X a : b Y c \rangle \quad (5.162)$$

$$\text{Misc.} \quad X \cdot f = X/f^\circ \quad (5.164)$$

$$f \backslash X = f^\circ \cdot X \quad (5.165)$$

$$X/\perp = T \quad (5.166)$$

$$X/id = X \quad (5.167)$$

$$R \setminus (f^\circ \cdot S) = f \cdot R \setminus S \quad (5.168)$$

$$R \setminus \top \cdot S = ! \cdot R \setminus ! \cdot S \quad (5.169)$$

$$R / (S \cup P) = R / S \cap R / P \quad (5.170)$$

RELATION DIFFERENCE, IMPLICATION AND NEGATION

Universal- $(-)$ $X - R \subseteq Y \equiv X \subseteq Y \cup R \quad (5.140)$

Pointwise- \Rightarrow $b(R \Rightarrow S)a \equiv (b R a) \Rightarrow (b S a) \quad (5.149)$

Universal- \Rightarrow $R \cap X \subseteq Y \equiv X \subseteq (R \Rightarrow Y) \quad (5.150)$

Distribution $f^\circ \cdot (R \Rightarrow S) \cdot g = (f^\circ \cdot R \cdot g) \Rightarrow (f^\circ \cdot S \cdot g) \quad (5.156)$

Definition- \neg $\neg R = (R \Rightarrow \perp) \quad (5.---)$

Pointwise- \neg $b(\neg R)a \Leftrightarrow \neg(b R a) \quad (5.---)$

Complementation $R \cup \neg R = \top \quad (5.154)$

Difference versus implication $\top - R \subseteq R \Rightarrow \perp \quad (5.155)$

de Morgan $\neg(R \cup S) = (\neg R) \cap (\neg S) \quad (5.152)$

Schröder's rule $\neg Q \cdot S^\circ \subseteq \neg R \Leftrightarrow R^\circ \cdot \neg Q \subseteq \neg S \quad (5.153)$

MONOTONICITY

Composition $R \subseteq S \wedge T \subseteq U \Rightarrow R \cdot T \subseteq S \cdot U \quad (5.78)$

Converse $R \subseteq S \Rightarrow R^\circ \subseteq S^\circ \quad (5.79)$

Intersection $R \subseteq S \wedge U \subseteq V \Rightarrow R \cap U \subseteq S \cap V \quad (5.80)$

Union $R \subseteq S \wedge U \subseteq V \Rightarrow R \cup U \subseteq S \cup V \quad (5.81)$

ENDO-RELATION TAXONOMY

Reflexive $id \subseteq R \quad (5.84)$

Coreflexive $R \subseteq id \quad (5.85)$

Transitive $R \cdot R \subseteq R \quad (5.86)$

Symmetric $R \subseteq R^\circ (\equiv R = R^\circ) \quad (5.87)$

$$\text{Anti-symmetric} \quad R \cap R^\circ \subseteq id \quad (5.88)$$

$$\text{Irreflexive} \quad R \cap id = \perp \quad (5.89)$$

$$\text{Connected} \quad R \cup R^\circ = \top \quad (5.90)$$

 PAIRING ("SPLITS")

$$\text{Pairing-pointwise} \quad (a, b) \langle R, S \rangle c \Leftrightarrow a R c \wedge b S c \quad (5.101)$$

$$\text{Pairing-def} \quad \langle R, S \rangle = \pi_1^\circ \cdot R \cap \pi_2^\circ \cdot S \quad (5.102)$$

$$\text{Universal-pairing} \quad X \subseteq \langle R, S \rangle \Leftrightarrow \begin{cases} \pi_1 \cdot X \subseteq R \\ \pi_2 \cdot X \subseteq S \end{cases} \quad (5.103)$$

$$\begin{aligned} \text{Pairing-fusion} \quad \langle R, S \rangle \cdot T &= \langle R \cdot T, S \cdot T \rangle \\ &\Leftrightarrow R \cdot (\text{img } T) \subseteq R \vee S \cdot (\text{img } T) \subseteq S \end{aligned} \quad (5.104)$$

$$\text{Pairing-fusion (functions)} \quad \langle R, S \rangle \cdot f = \langle R \cdot f, S \cdot f \rangle \quad (5.105)$$

$$\text{Pairing and converse} \quad \langle R, S \rangle^\circ \cdot \langle X, Y \rangle = (R^\circ \cdot X) \cap (S^\circ \cdot Y) \quad (5.108)$$

$$\text{Kernel of pairing} \quad \ker \langle R, S \rangle = \ker R \cap \ker S \quad (5.111)$$

$$\text{Functor-}\times\text{ def} \quad R \times S = \langle R \cdot \pi_1, S \cdot \pi_2 \rangle \quad (5.107)$$

$$\text{Functor-}\times\text{-id} \quad id \times id = id \quad (5.112)$$

$$\text{Functor-}\times\text{-composition} \quad (R \times S) \cdot (P \times Q) = (R \cdot P) \times (S \cdot Q) \quad (5.113)$$

$$\text{Functor-}\times\text{ absorption} \quad (R \times S) \cdot \langle P, Q \rangle = \langle R \cdot P, S \cdot Q \rangle \quad (5.106)$$

$$\text{Functor-}\times\text{-division} \quad \frac{f}{g} \times \frac{h}{k} = \frac{f \times h}{g \times k} \quad (5.127)$$

$$\text{Pairing division} \quad \frac{f}{g} \cap \frac{h}{k} = \frac{\langle f, h \rangle}{\langle g, k \rangle} \quad (5.109)$$

 COPRODUCTS

$$\text{Universal} \quad X = [R, S] \Leftrightarrow \begin{cases} X \cdot i_1 = R \\ X \cdot i_2 = S \end{cases} \quad (5.114)$$

$$\text{Definition} \quad [R, S] = R \cdot i_1^\circ \cup S \cdot i_2^\circ \quad (5.117)$$

$$\text{Reflexion} \quad \text{img } i_1 \cup \text{img } i_2 = id \quad (5.115)$$

$$\text{Disjointness} \quad i_1^\circ \cdot i_2 = \perp \quad (5.116)$$

$$\text{Either and converse} \quad [R, S] \cdot [T, U]^\circ = (R \cdot T^\circ) \cup (S \cdot U^\circ) \quad (5.121)$$

$$\text{Image of either} \quad \text{img } [R, S] = \text{img } R \cup \text{img } S \quad (5.124)$$

$$\text{Exchange law} \quad [\langle R, S \rangle, \langle T, V \rangle] = \langle [R, T], [S, V] \rangle \quad (5.122)$$

$$\mathbf{Functor}+-\mathbf{def} \quad R + S = [i_1 \cdot R, i_2 \cdot S] \quad (5.119)$$

$$\mathbf{Functor}+-\mathbf{converse} \quad (R + S)^\circ = R^\circ + S^\circ \quad (5.123)$$

$$\mathbf{Functor}+-\mathbf{division} \quad \frac{f}{g} + \frac{h}{k} = \frac{f+h}{g+k} \quad (5.128)$$

INJECTIVITY PREORDER

$$\mathbf{Definition} \quad R \leq S \equiv \ker S \subseteq \ker R \quad (5.234)$$

$$\mathbf{Join} \quad \langle R, S \rangle \leq X \equiv R \leq X \wedge S \leq X \quad (5.235)$$

$$\mathbf{Cancellation} \quad R \leq \langle R, S \rangle \quad \text{and} \quad S \leq \langle R, S \rangle. \quad (5.236)$$

$$\mathbf{Shunting} \quad R \cdot g \leq S \equiv R \leq S \cdot g^\circ \quad (5.237)$$

$$\mathbf{Meet} \quad X \leq [R^\circ, S^\circ]^\circ \Leftrightarrow X \leq R \wedge X \leq S \quad (5.238)$$

$$\mathbf{Either-injective} \quad id \leq [R, S] \Leftrightarrow id \leq R \wedge id \leq S \wedge R^\circ \cdot S = \perp \quad (5.126)$$

THUMB RULES

$$\text{A function } f \text{ is a bijection iff its converse } f^\circ \text{ is a function } g \quad (5.38)$$

$$\text{- converse of } \textit{injective} \text{ is } \textit{simple} \text{ (and vice-versa)} \quad (5.43)$$

$$\text{- converse of } \textit{entire} \text{ is } \textit{surjective} \text{ (and vice-versa)} \quad (5.44)$$

$$\text{- } \textit{smaller} \text{ than } \textit{injective} \text{ (simple) is } \textit{injective} \text{ (simple)} \quad (5.82)$$

$$\text{- } \textit{larger} \text{ than } \textit{entire} \text{ (surjective) is } \textit{entire} \text{ (surjective)} \quad (5.83)$$

$$\langle R, id \rangle \text{ is always } \textit{injective}, \text{ for whatever } R \quad (5.111a)$$

QUANTIFIER CALCULUS

$$\mathbf{Trading}-\forall \quad \langle \forall k : R \wedge S : T \rangle = \langle \forall k : R : S \Rightarrow T \rangle \quad (A.1)$$

$$\mathbf{Trading}-\exists \quad \langle \exists k : R \wedge S : T \rangle = \langle \exists k : R : S \wedge T \rangle \quad (A.2)$$

$$\mathbf{de Morgan} \quad \neg \langle \forall k : R : T \rangle = \langle \exists k : R : \neg T \rangle \quad (A.3)$$

$$\mathbf{de Morgan} \quad \neg \langle \exists k : R : T \rangle = \langle \forall k : R : \neg T \rangle \quad (A.4)$$

$$\mathbf{One-point}-\forall \quad \langle \forall k : k = e : T \rangle = T[k := e] \quad (A.5)$$

$$\mathbf{One-point}-\exists \quad \langle \exists k : k = e : T \rangle = T[k := e] \quad (A.6)$$

$$\mathbf{Nesting}-\forall \quad \langle \forall a, b : R \wedge S : T \rangle = \langle \forall a : R : \langle \forall b : S : T \rangle \rangle \quad (A.7)$$

$$\mathbf{Nesting-}\exists \quad \langle \exists a, b : R \wedge S : T \rangle = \langle \exists a : R : \langle \exists b : S : T \rangle \rangle \quad (\text{A.8})$$

$$\mathbf{Rearranging-}\forall \quad \langle \forall k : R \vee S : T \rangle = \langle \forall k : R : T \rangle \wedge \langle \forall k : S : T \rangle \quad (\text{A.9})$$

$$\mathbf{Rearranging-}\forall \quad \langle \forall k : R : T \wedge S \rangle = \langle \forall k : R : T \rangle \wedge \langle \forall k : R : S \rangle \quad (\text{A.10})$$

$$\mathbf{Rearranging-}\exists \quad \langle \exists k : R : T \vee S \rangle = \langle \exists k : R : T \rangle \vee \langle \exists k : R : S \rangle \quad (\text{A.11})$$

$$\mathbf{Rearranging-}\exists \quad \langle \exists k : R \vee S : T \rangle = \langle \exists k : R : T \rangle \vee \langle \exists k : S : T \rangle \quad (\text{A.12})$$

$$\mathbf{Splitting-}\forall \quad \langle \forall j : R : \langle \forall k : S : T \rangle \rangle = \langle \forall k : \langle \exists j : R : S \rangle : T \rangle \quad (\text{A.13})$$

$$\mathbf{Splitting-}\exists \quad \langle \exists j : R : \langle \exists k : S : T \rangle \rangle = \langle \exists k : \langle \exists j : R : S \rangle : T \rangle \quad (\text{A.14})$$