Closure, Properties and Closure Properties of Multirelations

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1. Multirelations
2. Reflexive-Transitive Closure
3. Properties and their Closure
4. Topological Contact

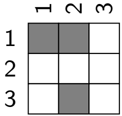
Context and Method

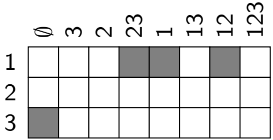
- multirelations in program semantics, games, topological contact
- systematically investigate their properties

- express multirelational operations using relations
- study properties of operations
- abstract properties to weak algebras
- derive theory in these algebras

Relations and Multirelations

- state space $A = \{1, 2, 3\}$
- relation $\subseteq A \times A$ multirelation $\subseteq A \times 2^A$





- Boolean algebra with $\cup$, $\cap$, $\overline{}$
- composition
- converse $\cdot^{\mathsf{c}}$, dual $\cdot^{\mathsf{d}}$

Multirelational Constants

O =

	$\emptyset$	3	2	23	1	13	12	123
1								
2								
3								

T =

	$\emptyset$	3	2	23	1	13	12	123
1	■	■	■	■	■	■	■	■
2	■	■	■	■	■	■	■	■
3	■	■	■	■	■	■	■	■

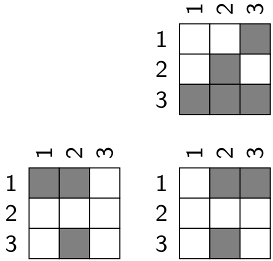
E =

	$\emptyset$	3	2	23	1	13	12	123
1					■	■	■	■
2			■	■			■	■
3		■		■		■		■

U =

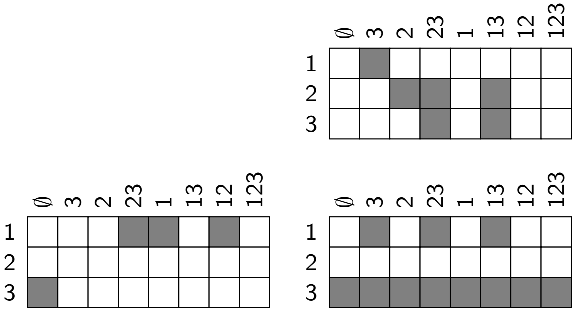
	$\emptyset$	3	2	23	1	13	12	123
1					■			
2			■					
3		■						

Relational Composition



$$(QR)_{x,z} \;\Leftrightarrow\; \exists y \in A : Q_{x,y} \wedge R_{y,z}$$

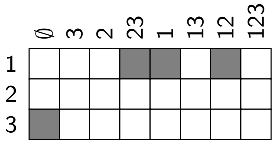
Multirelational Composition



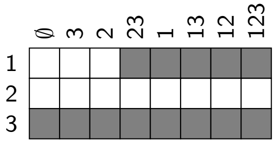
$$(Q\,;R)_{x,Z} \;\Leftrightarrow\; \exists Y \in 2^{A} : Q_{x,Y} \wedge \forall y \in Y : R_{y,Z}$$

Up-closed Multirelations

not up-closed



up-closed



$$\forall x \in A : \forall Y, Z \in 2^A : R_{x,Y} \wedge Y \subseteq Z \Rightarrow R_{x,Z}$$

Relational Operations for Multirelations

right residual	$Q \setminus R$	$=$	$\overline{Q^{\mathsf{c}}\overline{R}}$
symmetric quotient	$Q \div R$	$=$	$(Q \setminus R) \cap (R \setminus Q)^{\mathsf{c}}$
subset relation : $2^A \leftrightarrow 2^A$	S	$=$	$\mathsf{E} \setminus \mathsf{E}$
multirelational composition	$Q\,;R$	$=$	$Q(\mathsf{E} \setminus R)$
R up-closed if	R	$=$	$R\mathsf{S}$

Unit and Zero of Multirelations

left unit	$\mathsf{E}\,;R = \mathsf{E}(\mathsf{E} \setminus R) = R$
right unit	$R\,;\mathsf{E} = R(\mathsf{E} \setminus \mathsf{E}) = R\mathsf{S} = R$ if R up-closed
left zero	$\mathsf{O}\,;R = \mathsf{O}$
	$\mathsf{T}\,;R = \mathsf{T}$

Laws of Multirelations

all multirelations	up-closed multirelations
$\mathsf{O} ; R = \mathsf{O}$	
$\mathsf{E} ; R = R$	
$\mathsf{T} ; R = \mathsf{T}$	
$R ; \mathsf{E} \supseteq R$	$R ; \mathsf{E} = R$
$Q \subseteq R \Rightarrow P ; Q \subseteq P ; R$	
$(P \cup Q) ; R = P ; R \cup Q ; R$	
$(P \cap Q) ; R \subseteq P ; R \cap Q ; R$	$(P \cap Q) ; R = P ; R \cap Q ; R$
$(P ; Q) ; R \subseteq P ; (Q ; R)$	$(P ; Q) ; R = P ; (Q ; R)$

Algebraic Structures

bounded join-semilattice

$$x + (y + z) = (x + y) + z \qquad x + x = x$$
$$x + y = y + x \qquad 0 + x = x$$

pre-left semiring

$$(x \cdot y) + (x \cdot z) \leq x \cdot (y + z) \qquad (x \cdot y) \cdot z \leq x \cdot (y \cdot z)$$
$$(x \cdot z) + (y \cdot z) = (x + y) \cdot z \qquad x \leq x \cdot 1$$
$$0 = 0 \cdot x \qquad x = 1 \cdot x$$

left residual

$$x \cdot y \leq z \;\Leftrightarrow\; x \leq z/y$$

Reflexive-Transitive Closure

recursion modelled by

$$\begin{aligned} f(x) &= 1 + x \cdot y \\ g(x) &= 1 + y \cdot x \\ h(x) &= 1 + y + x \cdot x \end{aligned}$$

least prefixpoint

$$f(\mu f) \leq \mu f \qquad\qquad f(x) \leq x \;\Rightarrow\; \mu f \leq x$$

if μf, μg, μh exist then

$$\mu f \leq \mu g = \mu h$$

Properties of Multirelations

up-closed	$R\,;\mathsf{E} = R$		
total	$R\,;\mathsf{T} = \mathsf{T}$	co-total	$R\,;\mathsf{O} = \mathsf{O}$
$\cup$-distributive	$R\,;(P \cup Q) = R\,;P \cup R\,;Q$		
$\cap$-distributive	$R\,;(P \cap Q) = R\,;P \cap R\,;Q$		
reflexive	$\mathsf{E} \subseteq R$	co-reflexive	$R \subseteq \mathsf{E}$
transitive	$R\,;R \subseteq R$	dense	$R \subseteq R\,;R$
idempotent	$R\,;R = R$		
contact	$R\,;R \cup \mathsf{E} = R$	kernel	$R\,;R \cap \mathsf{E} = R\,;\mathsf{E}$
test	$R\,;\mathsf{T} \cap \mathsf{E} = R$	co-test	$R\,;\mathsf{O} \cup \mathsf{E} = R$
vector	$R\,;\mathsf{T} = R$		

Algebraic Structures

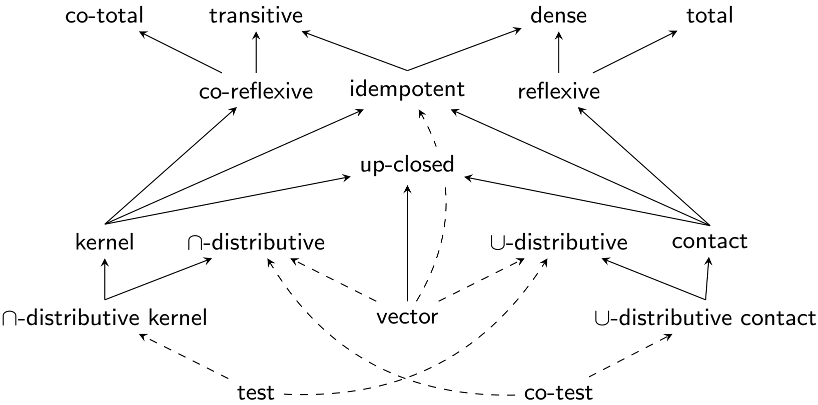
$(S, +, \curlywedge, 0, \top)$ bounded distributive lattice,
$(S, +, \cdot, 0, 1)$ pre-left semiring and

$$\begin{aligned} \top &= \top \cdot x \\ x \cdot (y \cdot z) &= (x \cdot (y \cdot 1)) \cdot z \\ (x \cdot z) \curlywedge (y \cdot z) &= ((x \cdot 1) \curlywedge (y \cdot 1)) \cdot z \end{aligned}$$

dual

$$\begin{aligned} (x \cdot y)^{\mathsf{d}} &= (x \cdot 1)^{\mathsf{d}} \cdot y^{\mathsf{d}} \\ (x + y)^{\mathsf{d}} &= x^{\mathsf{d}} \curlywedge y^{\mathsf{d}} \\ x^{\mathsf{dd}} &= x \\ 1^{\mathsf{d}} &= 1 \end{aligned}$$

Relationships between Properties



Closure Properties

	O	E	T	∪	∩	;	^{d}
total	–	■	■	■	□	□	▽
co-total	■	■	–	■	■	■	▲
transitive	■	■	■	–	■	–	▼
dense	■	■	■	■	–	–	△
reflexive	–	■	■	■	■	■	▼
co-reflexive	■	■	–	■	■	■	▲
idempotent	■	■	■	–	–	–	□
up-closed	■	■	■	■	■	■	■
∪-distributive	■	■	■	■	–	□	▽
∩-distributive	■	■	■	–	□	□	△
a contact	–	■	■	–	■	–	▼
a kernel	■	■	–	■	–	–	▲
a ∪-distributive contact	–	■	■	–	–	–	▼
a ∩-distributive kernel	■	■	–	–	–	–	▲
a test	■	■	–	■	■	■	▼
a co-test	–	■	■	■	■	■	▲
a vector	■	–	■	■	■	■	■

Topological Contact

- according to G. Aumann (1970)
- set of persons A
- set of topics T
- $t(x)$ = topics person x is interested in

$$t : A \to 2^T$$

- contact multirelation $R : A \leftrightarrow 2^A$

$$R_{x,Y} \;\Leftrightarrow\; t(x) \subseteq \bigcup_{y \in Y} t(y)$$

Axioms of Contact Relations

$$(K_0) \quad \neg\exists x \in A : R_{x,\emptyset}$$

$$(K_1) \quad \forall x \in A : R_{x,\{x\}}$$

$$(K_2) \quad \forall x \in A : \forall Y, Z \in 2^A : R_{x,Y} \wedge Y \subseteq Z \Rightarrow R_{x,Z}$$

$$(K_3) \quad \forall x \in A : \forall Y, Z \in 2^A : R_{x,Y} \wedge (\forall y \in Y : R_{y,Z}) \Rightarrow R_{x,Z}$$

$$(K_4) \quad \forall x \in A : \forall Y, Z \in 2^A : R_{x,Y \cup Z} \Leftrightarrow R_{x,Y} \vee R_{x,Z}$$

(K_1)–(K_3) contact relation

(K_0)–(K_4) topological contact relation

Examples of Topological Contact

- $\in$ $\qquad A \leftrightarrow 2^A$
- $R_{x,Y} \Leftrightarrow \exists y \in Y : f(x) = f(y)$ where $f : A \to B$
- $R_{x,Y} \Leftrightarrow \exists y \in Y : x \leq y$ $\qquad \mathbb{N} \leftrightarrow 2^{\mathbb{N}}$
- $R_{x,Y} \Leftrightarrow \exists y_1, y_2 \in Y : y_1 \leq x \leq y_2$
- $R_{x,Y} \Leftrightarrow \exists y_i \in Y : \exists r_i \in \mathbb{Q} : x = \sum r_i y_i$ $\qquad \mathbb{R}^n \leftrightarrow 2^{\mathbb{R}^n}$
- $R_{x,Y} \Leftrightarrow \exists y_i \in Y : \exists r_i \in \mathbb{Q}_0^+ : \sum r_i = 1 \land x = \sum r_i y_i$
- $R_{x,Y} \Leftrightarrow \forall \varepsilon > 0 : \exists y \in Y : d(x, y) < \varepsilon$

satisfy (K_0)–(K_3), some also (K_4)

Axioms using Multirelational Operations

(K_0)	$R ; \mathsf{O} = \mathsf{O}$	co-total
(K_1)	$\mathsf{E} \subseteq R$ (if R up-closed)	reflexive
(K_2)	$R ; \mathsf{E} = R$	up-closed
(K_3)	$R ; R \subseteq R$	transitive
(K_4)	$R ; (P \cup Q) = R ; P \cup R ; Q$ (if R up-closed)	$\cup$-distributive

Conclusion

- multirelations describe topological contact
- also consider not up-closed multirelations
- many results hold in weak algebras

- study connections to topology and closure systems
- generate further counterexamples
- give complete axioms