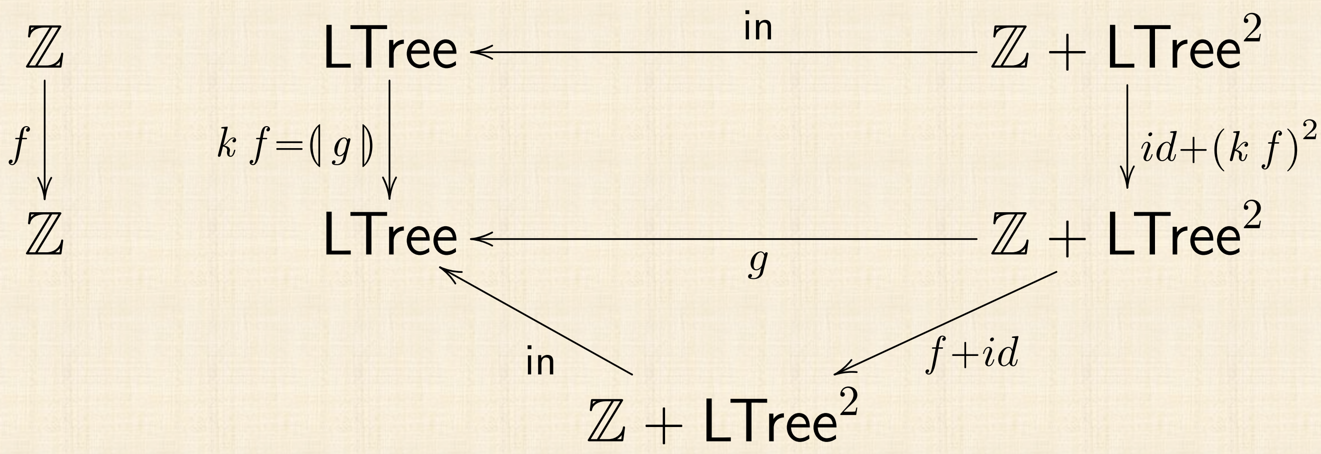
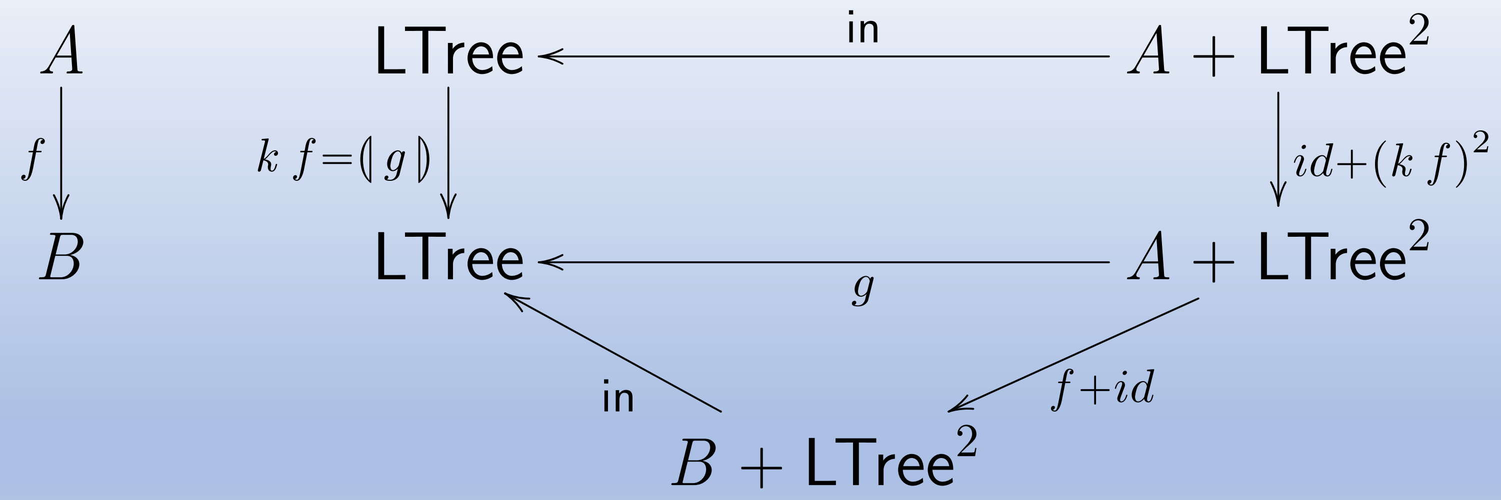
Cálculo de Programas
T09 (cntd)

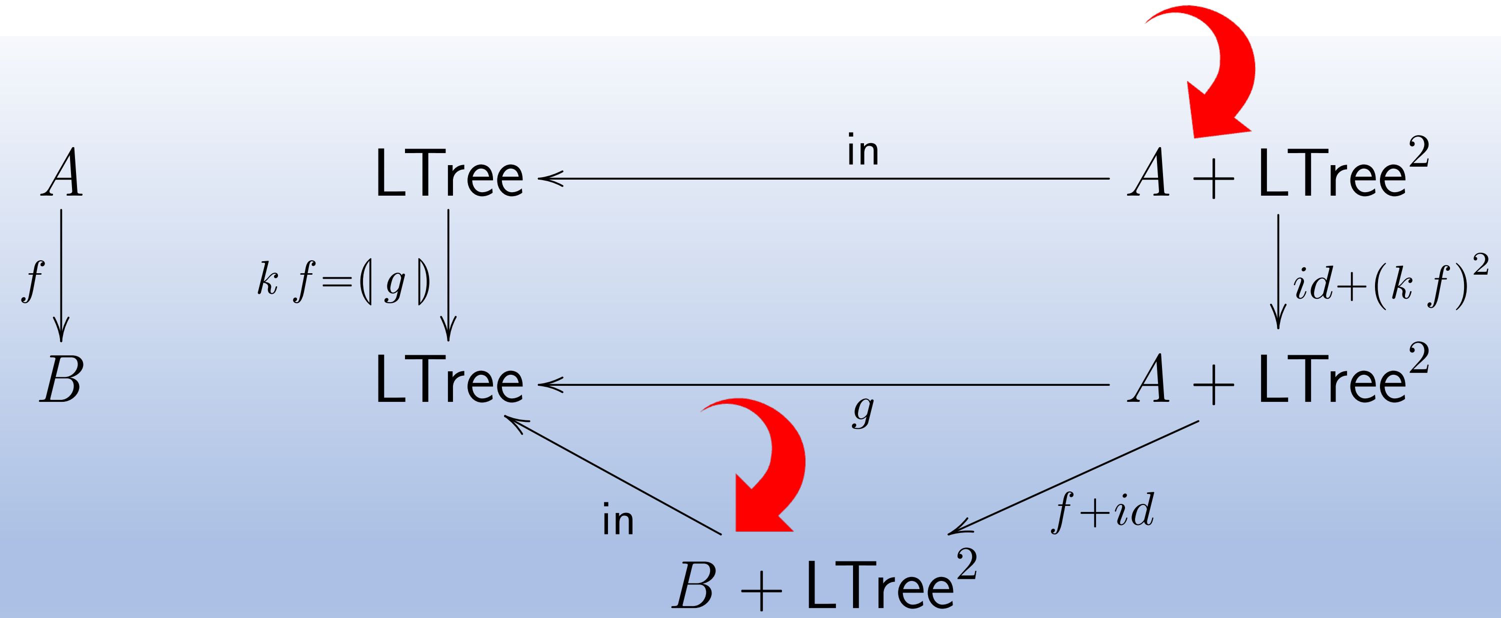
$$k\ f \cdot \mathsf{in} = \mathsf{in} \cdot (f + id) \cdot (id + k\ f \times k\ f)$$



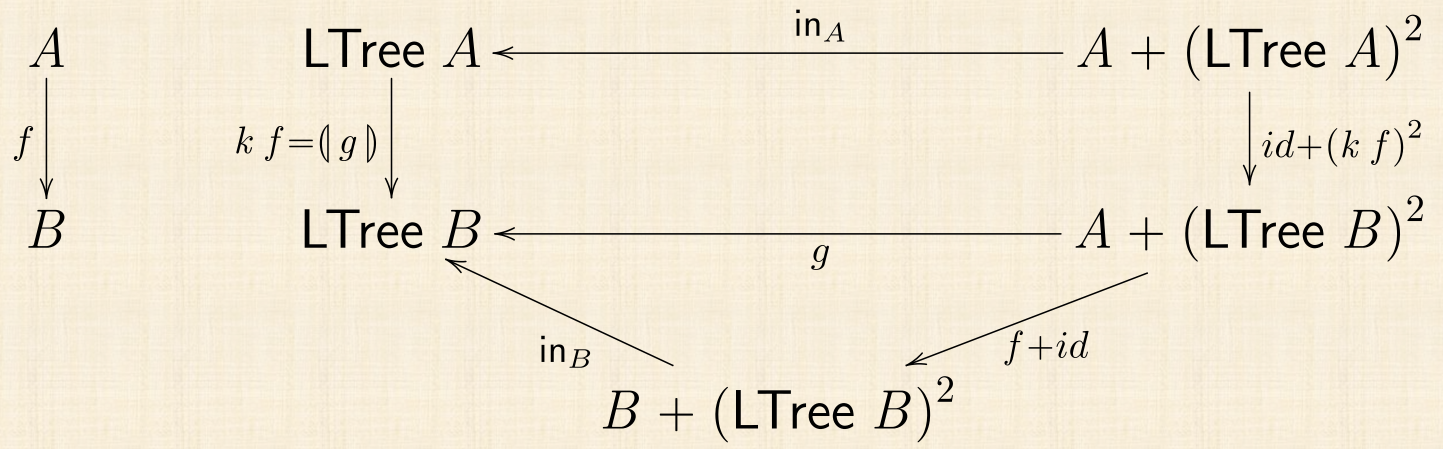
$$k\ f \cdot \mathsf{in} = \mathsf{in} \cdot (f + id) \cdot (id + k\ f \times k\ f)$$



$$k\ f \cdot \mathsf{in} = \mathsf{in} \cdot (f + id) \cdot (id + k\ f \times k\ f)$$



$$k\ f = (\!|\ \mathsf{in} \cdot (f + id)\ |\!)$$



$$k\ f = (\!|\ \mathsf{in} \cdot (f + id)\ |\!)$$

$$k\ f = (\!|\, \mathsf{in} \cdot (f + id)\, |\!)$$

$$k\ id = (\!|\, \mathsf{in}\, |\!) = id$$

Absorption:

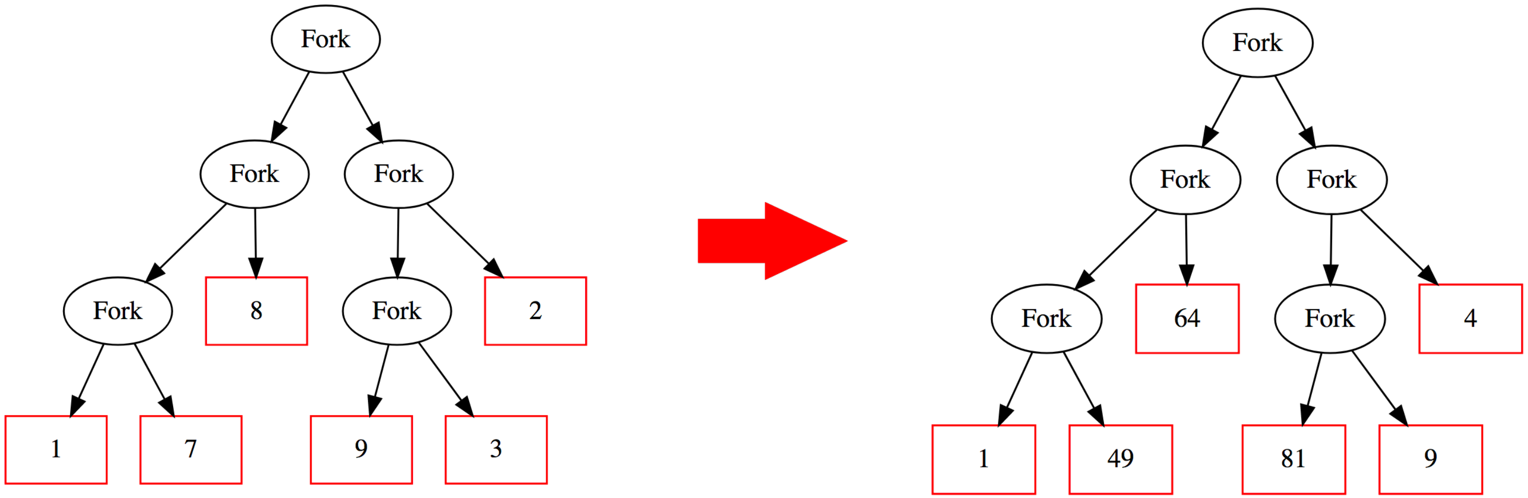
$$(\!|\, g\, |\!) \cdot k\ f = (\!|\, g \cdot (f + id)\, |\!)$$

$$
\begin{array}{ll}
& (\!| g |\!) \cdot k\ f = (\!| g \cdot (f + id) |\!) \\
\equiv & \quad \{\ k\ f = (\!| \mathsf{in} \cdot (f + id) |\!)\ \} \\
& (\!| g |\!) \cdot (\!| \mathsf{in} \cdot (f + id) |\!) = (\!| g \cdot (f + id) |\!) \\
\Leftarrow & \quad \{\ \text{cata-fusion}\ \} \\
& (\!| g |\!) \cdot \mathsf{in} \cdot (f + id) = g \cdot (f + id) \cdot (id + (\!| g |\!)^2) \\
\equiv & \quad \{\ \text{cata-cancellation}\ \} \\
& g \cdot (id + (\!| g |\!)^2) \cdot (f + id) = g \cdot (f + id) \cdot (id + (\!| g |\!)^2) \\
\equiv & \quad \{\ \text{functor-}+\text{ twice; natural-}id\text{ four times}\ \} \\
& g \cdot (f + (\!| g |\!)^2) = g \cdot (f + (\!| g |\!)^2) \\
\equiv & \quad \{\ \text{trivial}\ \} \\
& true
\end{array}
$$

$$
\begin{aligned}
& k\ (f \cdot g) \\
= & \quad \{\ k\ f = (\!| \mathsf{in} \cdot (f + id) |\!)\ \} \\
& (\!| \mathsf{in} \cdot (f \cdot g + id) |\!) \\
= & \quad \{\ \text{+-functor etc}\ \} \\
& (\!| \mathsf{in} \cdot (f + id) \cdot (g + id) |\!) \\
= & \quad \{\ \text{absorption (previous slides)}\ \} \\
& (\!| \mathsf{in} \cdot (f + id) |\!) \cdot (\!| \mathsf{in} \cdot (g + id) |\!) \\
= & \quad \{\ k\ f = (\!| \mathsf{in} \cdot (f + id) |\!)\ \text{twice}\ \} \\
& k\ f \cdot k\ g
\end{aligned}
$$

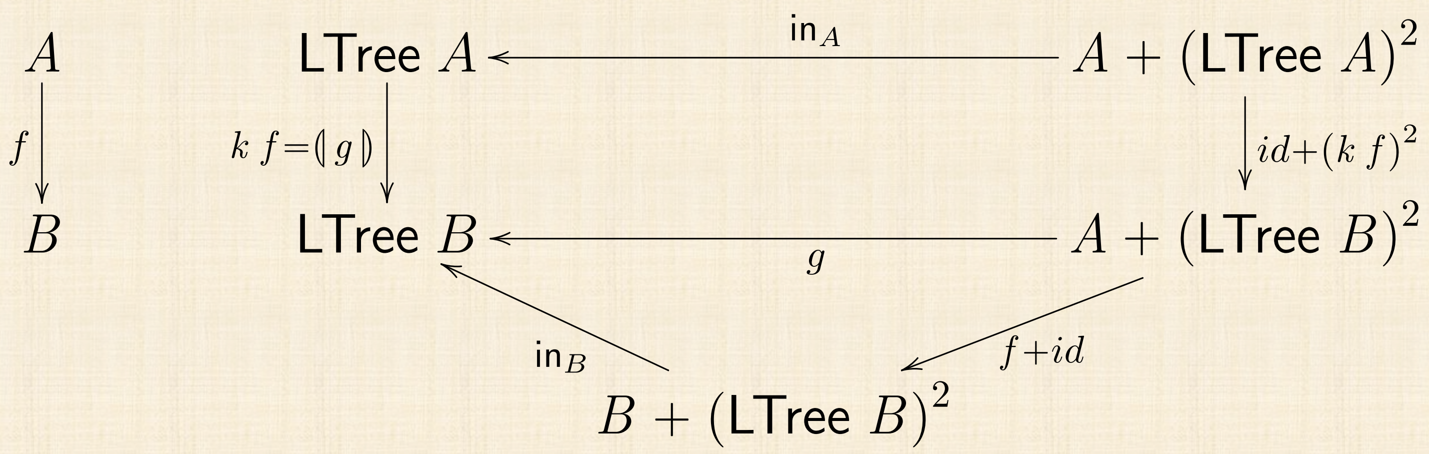
$$k\ id = (\!| \mathsf{in} |\!) = id$$

$$k\ (f \cdot g) = k\ f \cdot k\ g$$

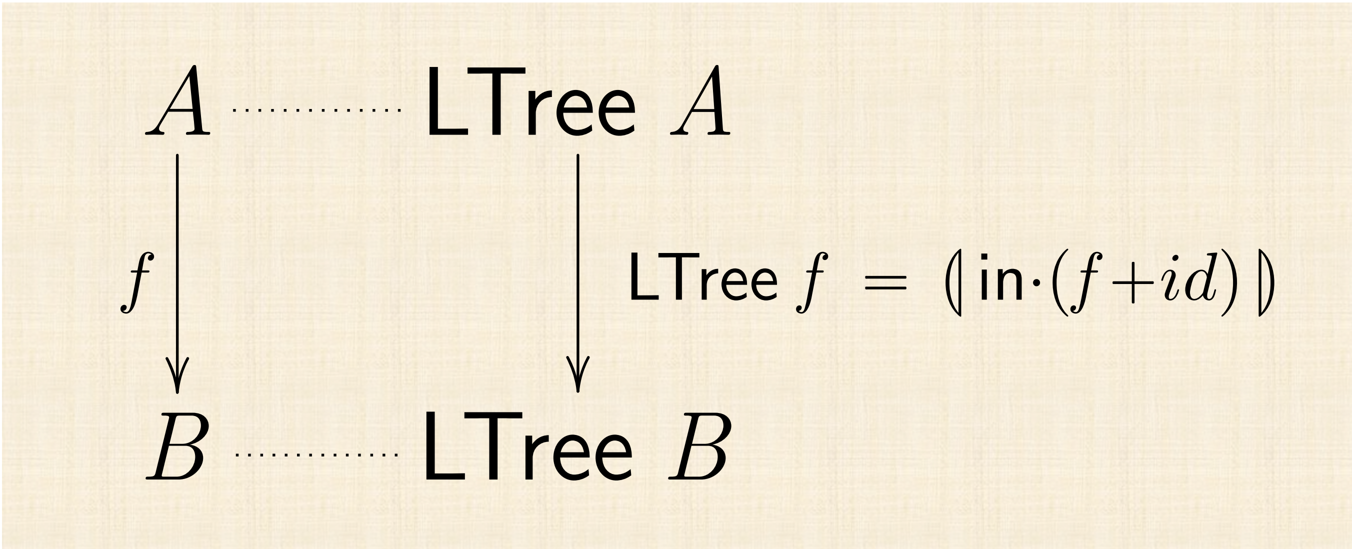



```
data LTree = Leaf Int | Fork (LTree, LTree)

k (Leaf x) = Leaf (x^2)
k (Fork (l,r)) = Fork (k l, k r)
```

$$k\ f = (\!|\, \mathsf{in} \cdot (f + id) \,|\!)$$



LTREE TYPE FUNCTOR



LTREE TYPE FUNCTOR

$$\mathsf{LTree}\ A \underset{\mathsf{in}}{\overset{\mathsf{out}}{\cong}} \underbrace{A + (\mathsf{LTree}\ A)^2}_{\mathbf{F}\ (\mathsf{LTree}\ A)}$$

$$\mathsf{LTree}\ B \underset{\mathsf{in}}{\overset{\mathsf{out}}{\cong}} \underbrace{B + (\mathsf{LTree}\ B)^2}_{\mathbf{F}\ (\mathsf{LTree}\ B)}$$

LTREE TYPE FUNCTOR

$$\mathbf{F}\ X = A + X^2 \qquad ?$$

$$\mathbf{F}\ X = B + X^2 \qquad ?$$

$$\mathsf{LTree}\ A \underset{\mathsf{in}}{\overset{\mathsf{out}}{\cong}} \underbrace{A + (\mathsf{LTree}\ A)^2}_{\mathbf{F}\ (\mathsf{LTree}\ A)}$$

$$\mathsf{LTree}\ B \underset{\mathsf{in}}{\overset{\mathsf{out}}{\cong}} \underbrace{B + (\mathsf{LTree}\ B)^2}_{\mathbf{F}\ (\mathsf{LTree}\ B)}$$

LTREE base bifunctor B

$$\mathsf{LTree}\ X \ \underset{\mathsf{in}}{\overset{\mathsf{out}}{\rightleftarrows}}\ \cong \ \underbrace{X + (\mathsf{LTree}\ X)^2}_{\mathbf{B}\ (X,\mathsf{LTree}\ X)}$$

$$\mathbf{B}\ (X,\, Y) = X + Y^2$$

BIFUNCTORS

$$
\begin{array}{ccccc}
A & \cdots\cdots & C & \cdots\cdots & \mathbf{B}\,(A, C) \\
{\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathbf{B}\,(f,g)} \\
D & \cdots\cdots & E & \cdots\cdots & \mathbf{B}\,(D, E)
\end{array}
$$

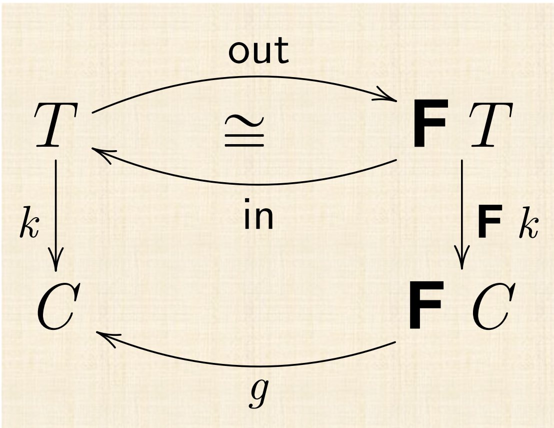
BIFUNCTORS (Laws)

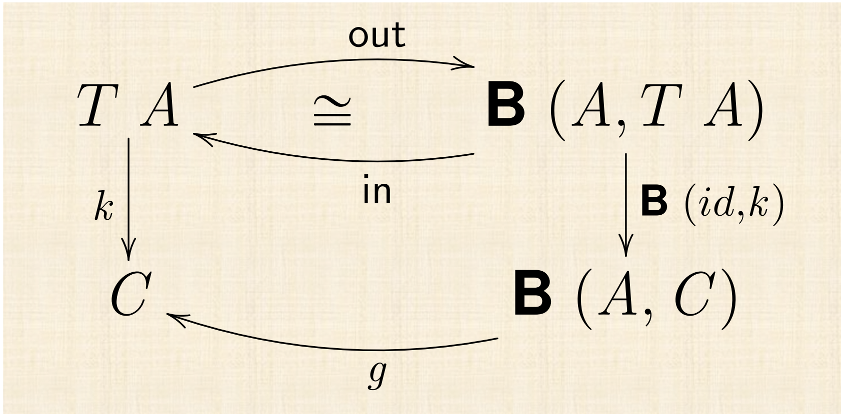
$$
\begin{array}{ccccc}
A & \cdots\cdots & C & \cdots\cdots & \mathbf{B}\,(A, C) \\
\downarrow f & & \downarrow g & & \downarrow \mathbf{B}\,(f, g) \\
D & \cdots\cdots & E & \cdots\cdots & \mathbf{B}\,(D, E)
\end{array}
$$

$$\mathbf{B}\,(id, id) = id$$

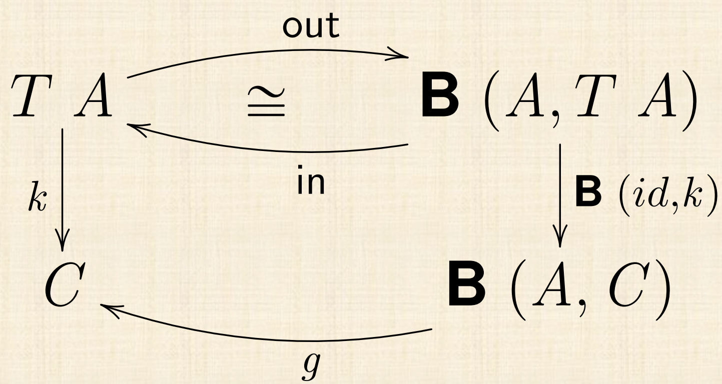
$$\mathbf{B}\,(h \cdot f, k \cdot g) = \mathbf{B}\,(h, k) \cdot \mathbf{B}\,(f, g)$$

CATAMORPHISMS (generalization)





CATAMORPHISMS (in general)



Universal property

$$k = (\!|\, g \,|\!) \Leftrightarrow k \cdot \mathsf{in} = g \cdot \mathbf{B}\ (id, k)$$

Type functor:

$$T\ f = (\!|\, \mathsf{in} \cdot \mathbf{B}\ (f, id) \,|\!)$$

Abbreviation:

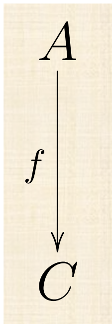
$$\mathbf{F}\ k = \mathbf{B}\ (id, k)$$

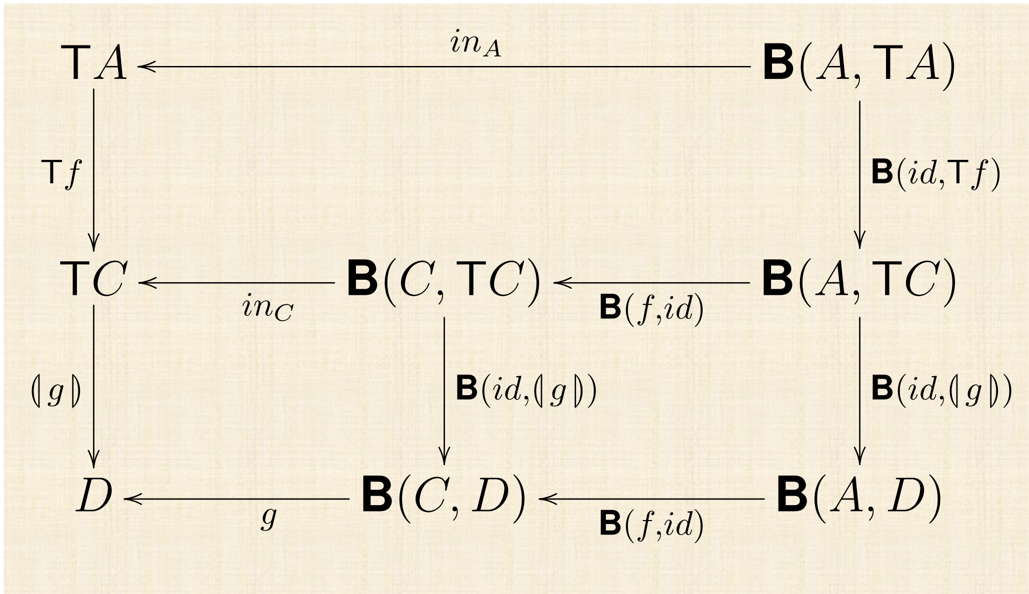
CATAMORPHISMS (Laws)

Universal-cata	$k = (\| g \|) \quad \Leftrightarrow \quad k \cdot \mathsf{in} = g \cdot \mathsf{F}\, k$	(43)
Cancelamento-cata	$(\| g \|) \cdot \mathsf{in} = g \cdot \mathsf{F}\, (\| g \|)$	(44)
Reflexão-cata	$(\| \mathsf{in} \|) = id_{\mathsf{T}}$	(45)
Fusão-cata	$f \cdot (\| g \|) = (\| h \|) \quad \Leftarrow \quad f \cdot g = h \cdot \mathsf{F}\, f$	(46)
Base-cata	$\mathsf{F}\, f \quad = \quad \mathsf{B}\, (id, f)$	(47)
Def-map-cata	$\mathsf{T}\, f \quad = \quad (\| \mathsf{in} \cdot \mathsf{B}(f, id) \|)$	(48)
Absorção-cata	$(\| g \|) \cdot \mathsf{T}\, f \quad = \quad (\| g \cdot \mathsf{B}(f, id) \|)$	(49)

Cata-absorption

$$(\!|\, g \,|\!) \cdot \mathsf{T} f = (\!|\, g \cdot \mathbf{B}\,(f, id) \,|\!)$$





(a) Trees whose data of type A are stored in their nodes:

$$\mathsf{T} = \mathsf{BTree}\ A \qquad \begin{cases} \mathsf{F}\ X = 1 + A \times X^2 \\ \mathsf{F}\ f = id + id \times f^2 \end{cases} \qquad \mathsf{in} = [\underline{Empty}\ , Node]$$

Haskell: **data** $\mathsf{BTree}\ a = Empty \mid Node\ (a, (\mathsf{BTree}\ a, \mathsf{BTree}\ a))$

(b) Trees with data in their leafs :

$$\mathsf{T} = \mathsf{LTree}\ A \qquad \begin{cases} \mathsf{F}\ X = A + X^2 \\ \mathsf{F}\ f = id + f^2 \end{cases} \qquad \mathsf{in} = [Leaf\ , Fork]$$

Haskell: **data** $\mathsf{LTree}\ a = Leaf\ a \mid Fork\ (\mathsf{LTree}\ a, \mathsf{LTree}\ a)$

(c) Full trees — data in both leaves and nodes:

$$\mathsf{T} = \mathsf{FTree}\ B\ A \qquad \begin{cases} \mathsf{F}\ X = B + A \times X^2 \\ \mathsf{F}\ f = id + id \times f^2 \end{cases} \qquad \mathsf{in} = [Unit\ , Comp]$$

Haskell: **data** $\mathsf{FTree}\ b\ a = Unit\ b \mid Comp\ (a, (\mathsf{FTree}\ b\ a, \mathsf{FTree}\ b\ a))$

(d) Expression trees:

$$\mathsf{T} = \mathsf{Expr}\ V\ O \qquad \begin{cases} \mathsf{F}\ X = V + O \times X^* \\ \mathsf{F}\ f = id + id \times \mathsf{map}\ f \end{cases} \qquad \mathsf{in} = [Var\ , Op]$$

Haskell: **data** $\mathsf{Expr}\ v\ o = Var\ v \mid Op\ (o, [\mathsf{Expr}\ v\ o])$

(a) Trees whose data of type A are stored in their nodes:

$$\mathsf{T} = \mathsf{BTree}\ A \qquad \begin{cases} \mathsf{B}\ (X,Y) = 1 + X \times Y^2 \\ \mathsf{B}\ (g,f) = id + g \times f^2 \end{cases} \qquad \mathsf{in} = [\underline{Empty}\ , Node]$$

Haskell: $\textbf{data}\ \mathsf{BTree}\ a = Empty \mid Node\ (a, (\mathsf{BTree}\ a, \mathsf{BTree}\ a))$

(b) Trees with data in their leafs :

$$\mathsf{T} = \mathsf{LTree}\ A \qquad \begin{cases} \mathsf{B}\ (X,Y) = X + Y^2 \\ \mathsf{B}\ (g,f) = g + f^2 \end{cases} \qquad \mathsf{in} = [Leaf\ , Fork]$$

Haskell: $\textbf{data}\ \mathsf{LTree}\ a = Leaf\ a \mid Fork\ (\mathsf{LTree}\ a, \mathsf{LTree}\ a)$

(c) Full trees — data in both leaves and nodes:

$$\mathsf{T} = \mathsf{FTree}\ B\ A \qquad \begin{cases} \mathsf{B}\ (Z,X,Y) = Z + X \times Y^2 \\ \mathsf{B}\ (h,g,f) = h + g \times f^2 \end{cases} \qquad \mathsf{in} = [Unit\ , Comp]$$

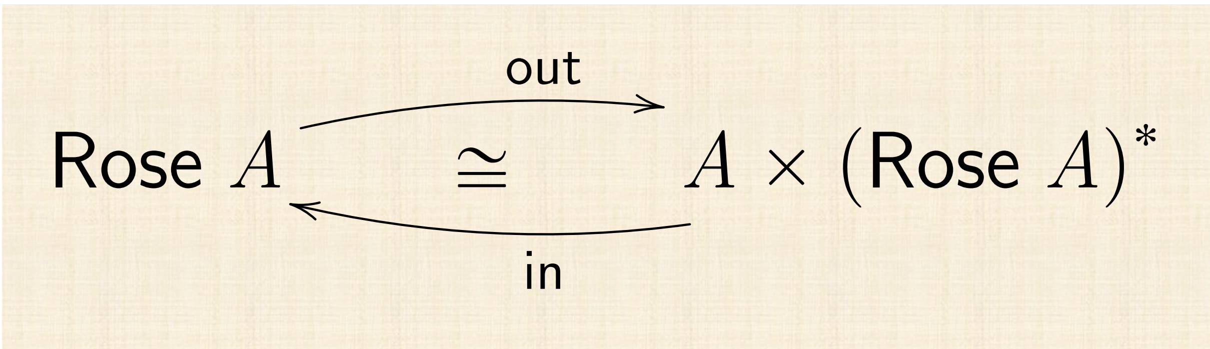
Haskell: $\textbf{data}\ \mathsf{FTree}\ b\ a = Unit\ b \mid Comp\ (a, (\mathsf{FTree}\ b\ a, \mathsf{FTree}\ b\ a))$

(d) Expression trees:

$$\mathsf{T} = \mathsf{Expr}\ V\ O \qquad \begin{cases} \mathsf{B}\ (Z,X,Y) = Z + X \times Y^* \\ \mathsf{B}\ (h,g,f) = h + g \times \mathsf{map}\ f \end{cases} \qquad \mathsf{in} = [Var\ , Op]$$

Haskell: $\textbf{data}\ \mathsf{Expr}\ v\ o = Var\ v \mid Op\ (o, [\mathsf{Expr}\ v\ o])$

```haskell
data Rose a = Rose a [Rose a] deriving Show

inRose = uncurry Rose

outRose (Rose a x) = (a,x)
```



$$\mathsf{Rose}\ A \ \underset{\mathsf{in}}{\overset{\mathsf{out}}{\cong}}\ A \times (\mathsf{Rose}\ A)^*$$

$$\mathbf{B}\,(X, Y) = X \times Y^*$$
$$\mathbf{B}\,(f, g) = f \times g^*$$

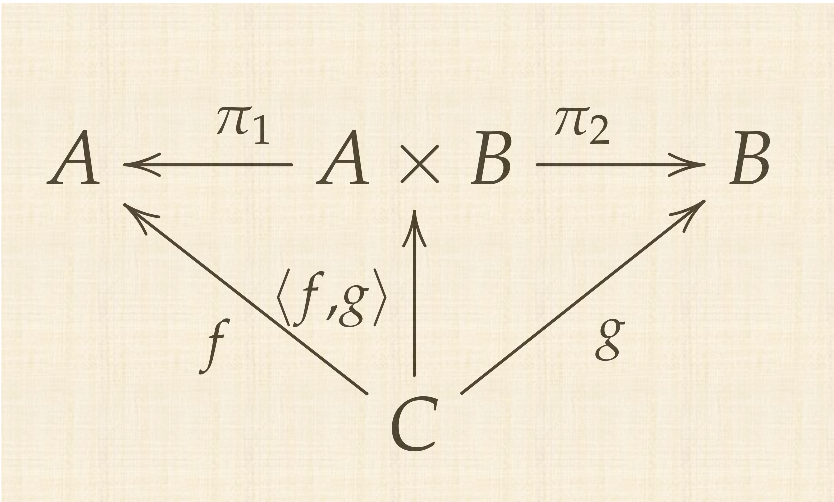
$$A \times Y^*$$
$$X \times Y^*$$

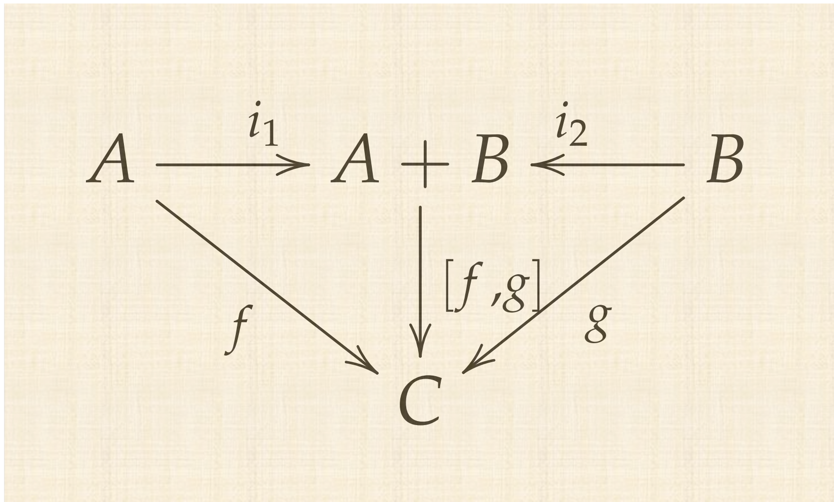
```haskell
data Rose a = Rose a [Rose a] deriving Show

inRose = uncurry Rose

outRose (Rose a x) = (a,x)
```

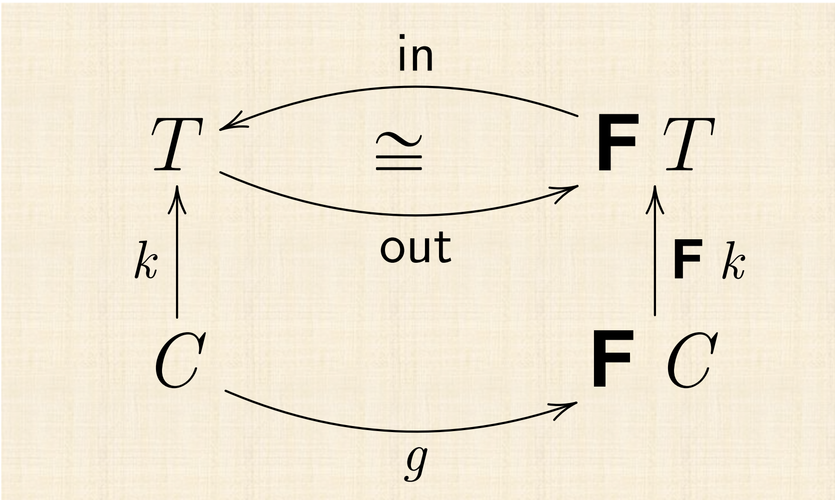
$$\mathbf{B}\,(X, Y) = X \times Y^*$$
$$\mathbf{B}\,(f, g) = f \times g^*$$

```haskell
f >< map g
```

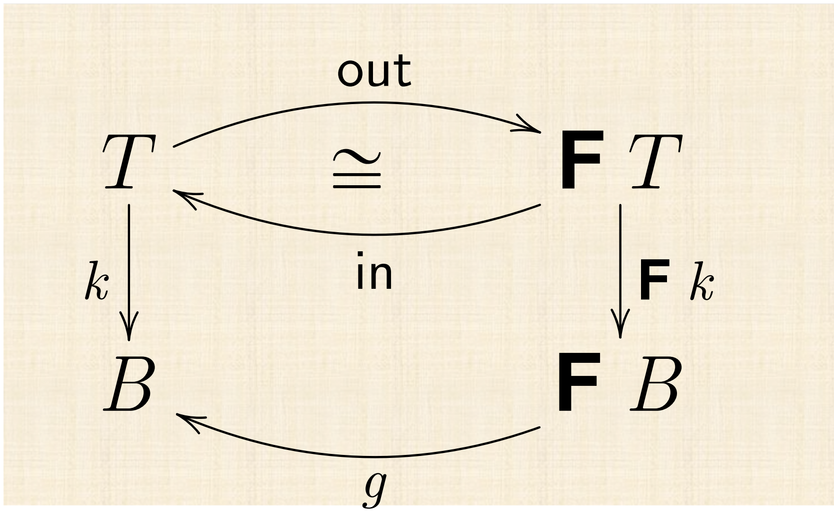
Anamorphisms


A
π1
A × B
π2
B
⟨f,g⟩
f
g
C


A
i1
A + B
i2
B
[f,g]
f
g
C

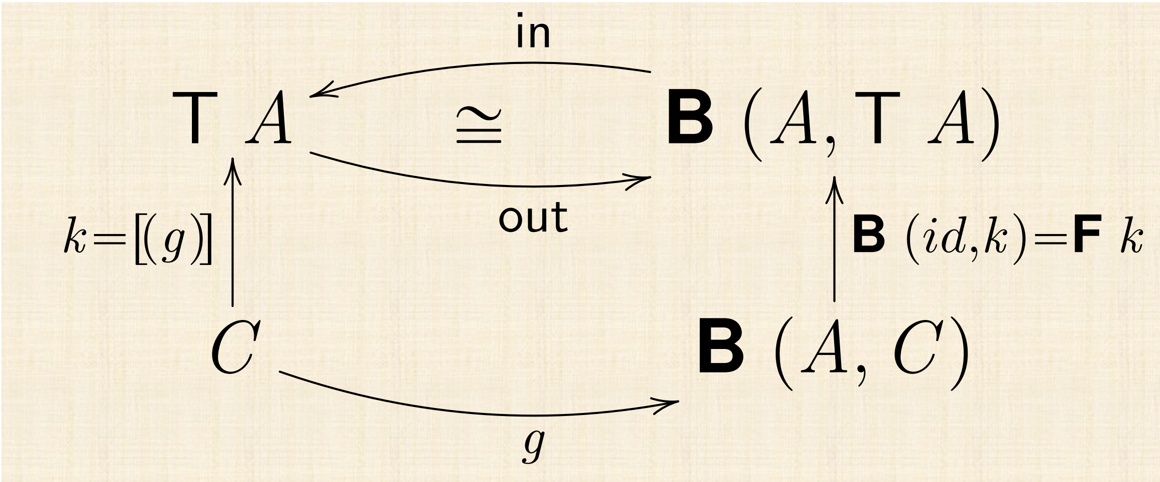


$\alpha\nu\alpha$ (ana)



$\kappa\alpha\tau\alpha$ (cata)

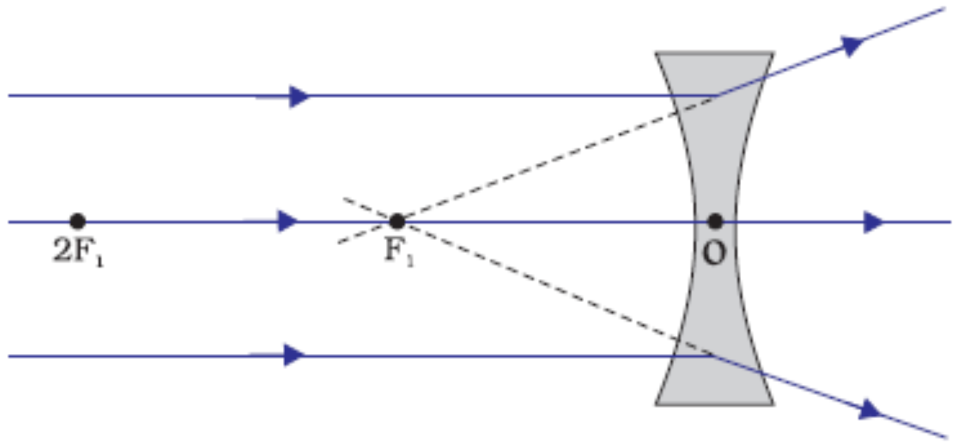
ANAMORPHISMS



$$k = [\!(g)\!] \;\Leftrightarrow\; \mathsf{in} \cdot \mathbf{F}\, k \cdot g$$

ANAMORPHISM LAWS

Universal-ana	$k = [\![(g)]\!] \quad \Leftrightarrow \quad \mathsf{out} \cdot k = (\mathsf{F}\, k) \cdot g$	(52)
Cancelamento-ana	$\mathsf{out} \cdot [\![(g)]\!] = \mathsf{F}\, [\![(g)]\!] \cdot g$	(53)
Reflexão-ana	$[\![(\mathsf{out})]\!] = id_{\mathsf{T}}$	(54)
Fusão-ana	$[\![(g)]\!] \cdot f = [\![(h)]\!] \quad \Leftarrow \quad g \cdot f = (\mathsf{F}\, f) \cdot h$	(55)
Base-ana	$\mathsf{F}\, f \quad = \quad \mathsf{B}\ (id, f)$	(56)
Def-map-ana	$\mathsf{T}\, f \quad = \quad [\![(\mathsf{B}(f, id) \cdot \mathsf{out})]\!]$	(57)
Absorção-ana	$\mathsf{T}\, f \cdot [\![(g)]\!] \quad = \quad [\![(\mathsf{B}(f, id) \cdot g)]\!]$	(58)


2F₁
F₁
O

Example

$$
\begin{array}{ccccc}
\mathbb{N}_0{}^* & \xleftarrow{\quad\mathsf{in}\quad} & & & 1+\mathbb{N}_0\times\mathbb{N}_0{}^* \\
\Big\uparrow{\scriptstyle k} & & & & \Big\uparrow{\scriptstyle id+id\times k} \\
\mathbb{N}_0 & \xrightarrow[\mathsf{out}_{\mathbb{N}_0}]{} & 1+\mathbb{N}_0 & \xrightarrow[id+\langle\mathsf{succ},id\rangle)]{} & 1+\mathbb{N}_0\times\mathbb{N}_0
\end{array}
$$

$$k = [\!((id + \langle \mathsf{succ}, id \rangle) \cdot \mathsf{out}_{\mathbb{N}_0})\!]$$

$$\equiv \qquad \{ \text{ ana-universal } \}$$

$$k = \mathsf{in} \cdot (id + id \times k) \cdot (id + \langle \mathsf{succ}, id \rangle) \cdot \mathsf{out}_{\mathbb{N}_0}$$

$$\equiv \qquad \{ \text{ isomorphism } \mathsf{in}_{\mathbb{N}_0} \, / \, \mathsf{out}_{\mathbb{N}_0} \}$$

$$k \cdot \mathsf{in}_{\mathbb{N}_0} = \mathsf{in} \cdot (id + id \times k) \cdot (id + \langle \mathsf{succ}, id \rangle)$$

$$\equiv \qquad \{ \text{ functor-}+\text{; absorption-}\times \}$$

$$k \cdot \mathsf{in}_{\mathbb{N}_0} = \mathsf{in} \cdot (id + \langle \mathsf{succ}, k \rangle)$$

$\equiv$ { definitions of in and $\mathsf{in}_{\mathbb{N}_0}$; fusion and absorption-$+$ }

$$[k \cdot \underline{0}, k \cdot \mathsf{succ}] = [\mathsf{nil}, \mathsf{cons} \cdot \langle \mathsf{succ}, k \rangle]$$

$\equiv$ { eq-$+$ }

$$\begin{cases} k \cdot \underline{0} = \mathsf{nil} \\ k \cdot \mathsf{succ} = \mathsf{cons} \cdot \langle \mathsf{succ}, k \rangle \end{cases}$$

$\equiv$ { going pointwise }

$$\begin{cases} k\ 0 = [\,] \\ k\ (n+1) = (n+1) : k\ n \end{cases}$$

$$
\begin{array}{ccccc}
{\mathbb{N}_0}^* & & \xleftarrow{\quad\quad\mathsf{in}\quad\quad} & & 1 + \mathbb{N}_0 \times {\mathbb{N}_0}^* \\
\uparrow{\scriptstyle k} & & & & \uparrow{\scriptstyle id+id\times k} \\
\mathbb{N}_0 & \xrightarrow[\mathsf{out}_{\mathbb{N}_0}]{} & 1 + \mathbb{N}_0 & \xrightarrow[id+\langle \mathsf{succ},id\rangle)]{} & 1 + \mathbb{N}_0 \times \mathbb{N}_0
\end{array}
$$

$$
\begin{array}{ccccc}
\mathbb{N}_0 & & \xleftarrow{\quad[\underline{1},\mathsf{mul}]\quad} & & 1+\mathbb{N}_0\times\mathbb{N}_0 \\
\uparrow{\scriptstyle m} & & & & \uparrow{\scriptstyle id+id\times m} \\
{\mathbb{N}_0}^* & & \xleftarrow{\quad\mathsf{in}\quad} & & 1+\mathbb{N}_0\times{\mathbb{N}_0}^* \\
\uparrow{\scriptstyle k} & & & & \uparrow{\scriptstyle id+id\times k} \\
\mathbb{N}_0 & \xrightarrow[\mathsf{out}_{\mathbb{N}_0}]{} & 1+\mathbb{N}_0 & \xrightarrow[id+\langle\mathsf{succ},id\rangle)]{} & 1+\mathbb{N}_0\times\mathbb{N}_0
\end{array}
$$

$$f = m \cdot k$$

$$\equiv \quad \{\ m = (\![\underline{1}, \mathsf{mul}]\!) \text{ e } k = [\!((id + \langle \mathsf{succ}, id\rangle) \cdot \mathsf{out}_{\mathbb{N}_0})\!] \ \}$$

$$f = (\![\underline{1}, \mathsf{mul}]\!) \cdot [\!((id + \langle \mathsf{succ}, id\rangle) \cdot \mathsf{out}_{\mathbb{N}_0})\!]$$

$$\equiv \quad \{\ \text{cancellations (ana and cata)}\ \}$$

$$f = [\underline{1}, \mathsf{mul}] \cdot \mathbf{F}\ m \cdot \mathsf{out} \cdot \mathsf{in} \cdot \mathbf{F}\ k \cdot (id + \langle \mathsf{succ}, id\rangle) \cdot \mathsf{out}_{\mathbb{N}_0}$$

$$\equiv \quad \{\ \mathsf{in} \cdot \mathsf{out} = id\ ;\ \text{functor } \mathbf{F}\text{: } (\mathbf{F}\ m) \cdot (\mathbf{F}\ k) = \mathbf{F}\ (m \cdot k)\ \}$$

$$f = [\underline{1}, \mathsf{mul}] \cdot \mathbf{F}\ (m \cdot k) \cdot (id + \langle \mathsf{succ}, id\rangle) \cdot \mathsf{out}_{\mathbb{N}_0}$$

$$\equiv \quad \{\ \text{isomorphism } \mathsf{in}_{\mathbb{N}_0} / \mathsf{out}_{\mathbb{N}_0};\ m \cdot k = f\ ;\ \mathbf{F}\ f = id + id \times f\ \}$$

$$f \cdot \mathsf{in}_{\mathbb{N}_0} = [\underline{1}, \mathsf{mul}] \cdot (id + id \times f) \cdot (id + \langle \mathsf{succ}, id\rangle)$$

$$f \cdot \mathsf{in}_{\mathbb{N}_0} = [\underline{1}, \mathsf{mul}] \cdot (id + id \times f) \cdot (id + \langle \mathsf{succ}, id \rangle)$$

$\equiv$ { +-absorption ; $\times$-absorption ; etc }

$$f \cdot \mathsf{in}_{\mathbb{N}_0} = [\underline{1}, \mathsf{mul} \cdot \langle \mathsf{succ}, f \rangle]$$

$\equiv$ { Eq-+ ; $\mathsf{in}_{\mathbb{N}_0} = [\underline{0}, \mathsf{succ}]$ }

$$\begin{cases} f \cdot \underline{0} = \underline{1} \\ f \cdot \mathsf{succ} = \mathsf{mul} \cdot \langle \mathsf{succ}, f \rangle \end{cases}$$

$\equiv$ { go pointwise }

$$\begin{cases} f\ 0 = 1 \\ f\ (n+1) = (n+1) \times f\ n \end{cases}$$

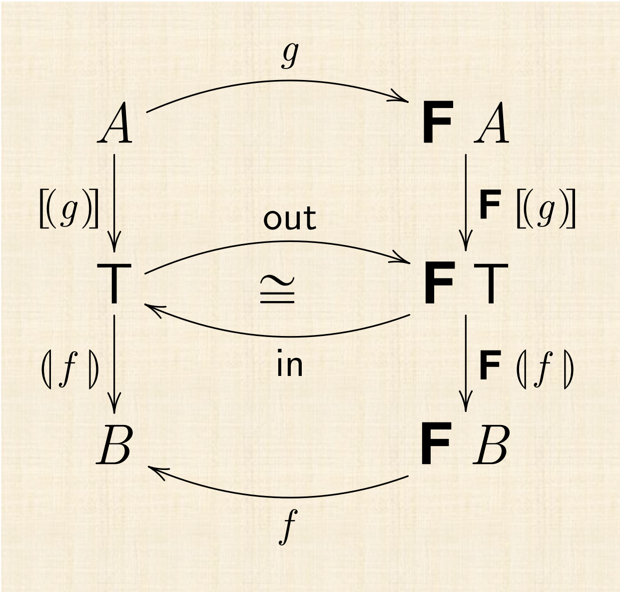
HILOMORPHISM

"Hylo + morphism"

ξύλο = matter, thing

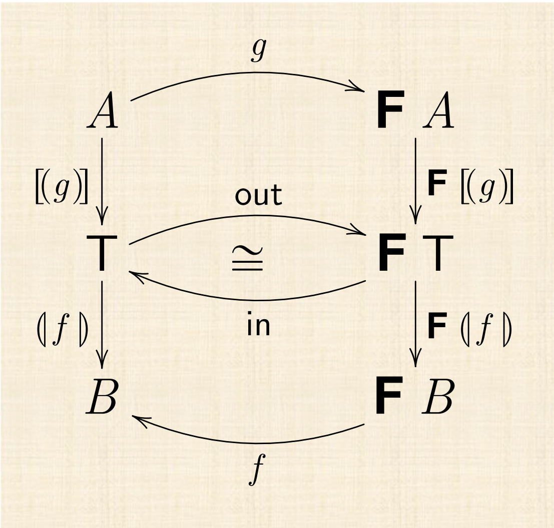
$$[\![f, g]\!] = (\!| f |\!) \cdot [\!(g)\!]$$

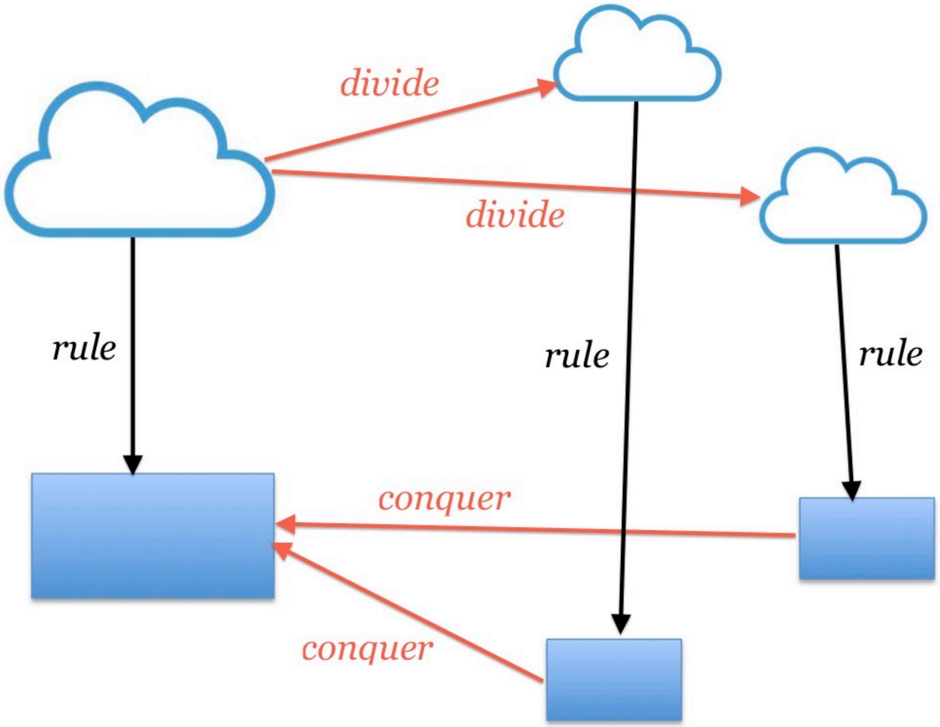
HILOMORPHISM



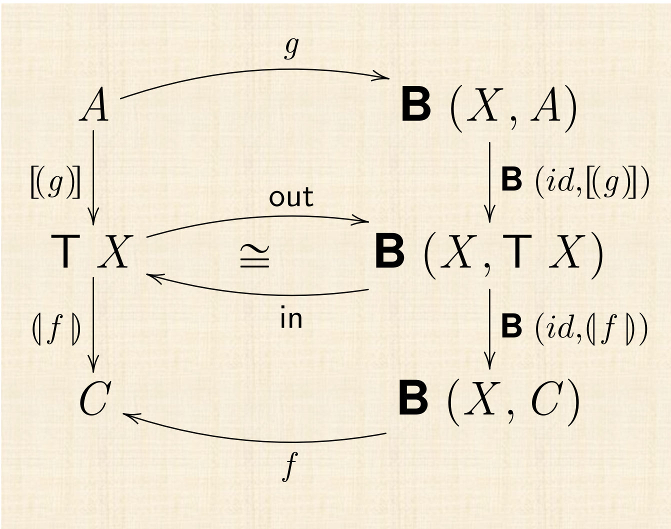
$$[\![f, g]\!] = (\!| f |\!) \cdot [\!(g)\!]$$

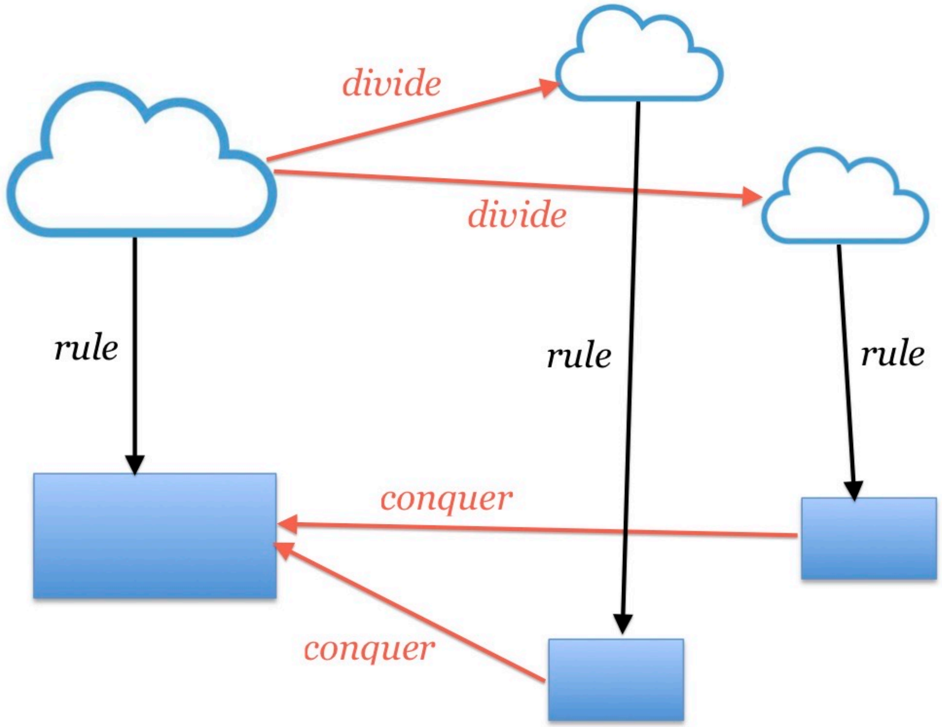
'DIVIDE & CONQUER'



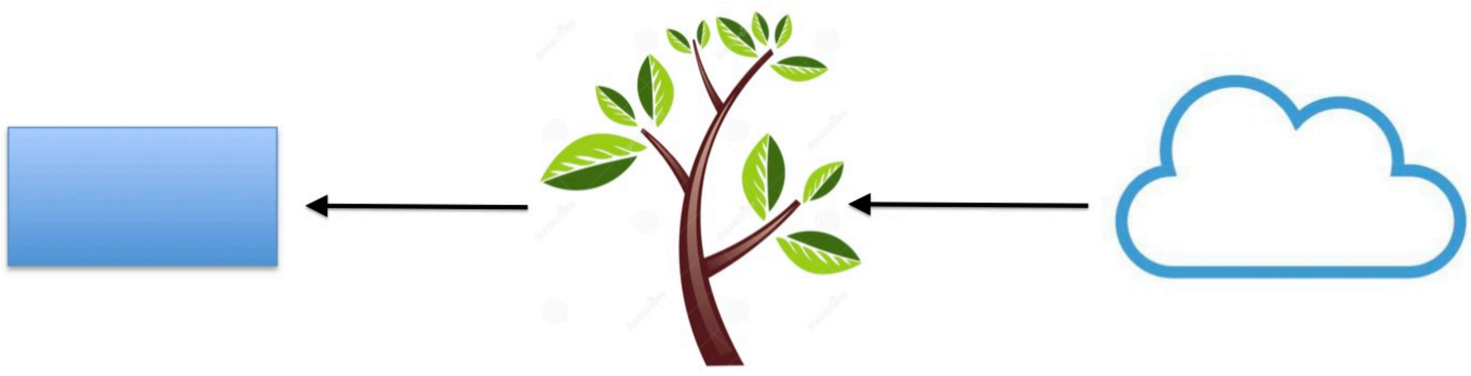


'DIVIDE & CONQUER'



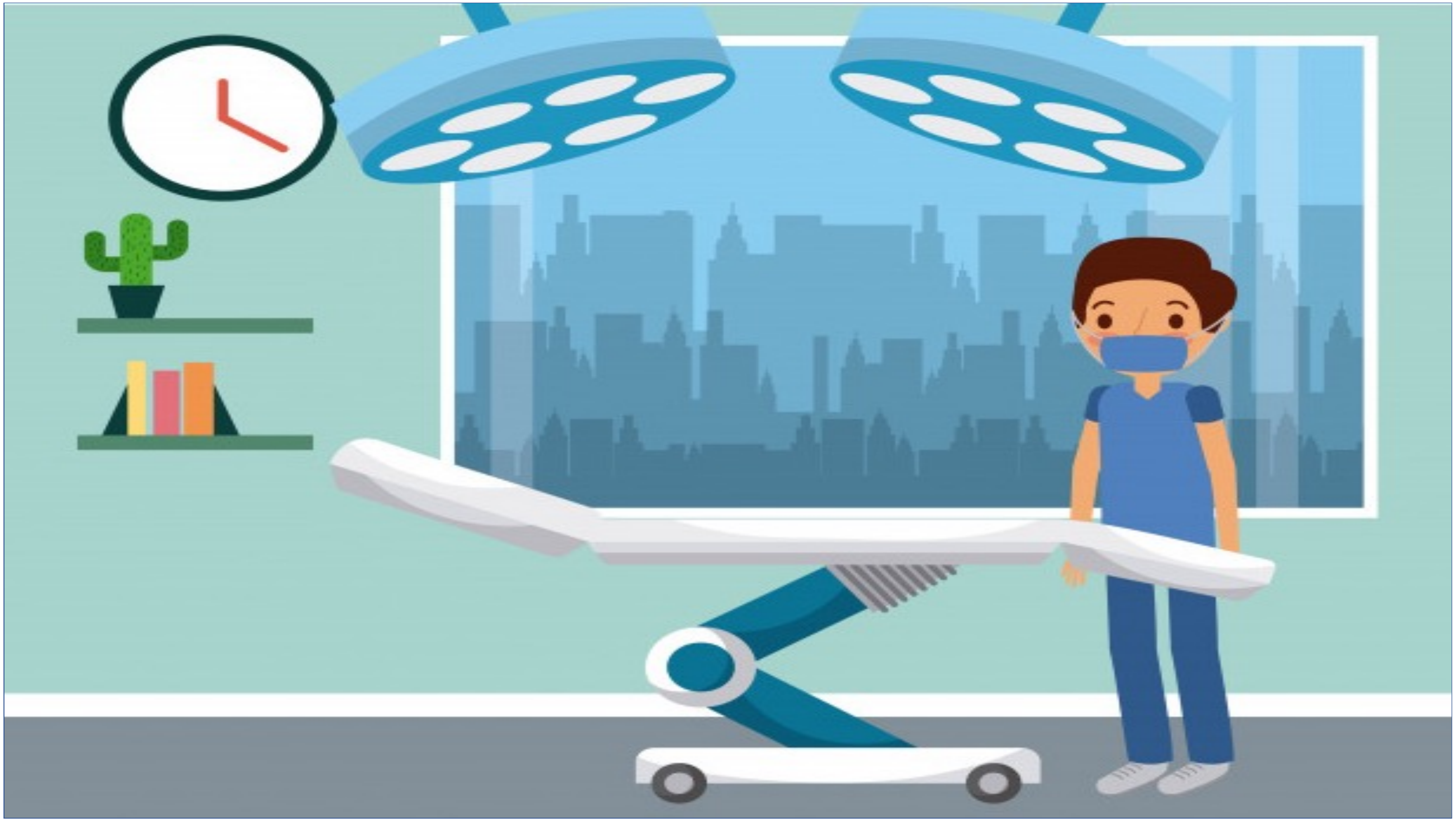


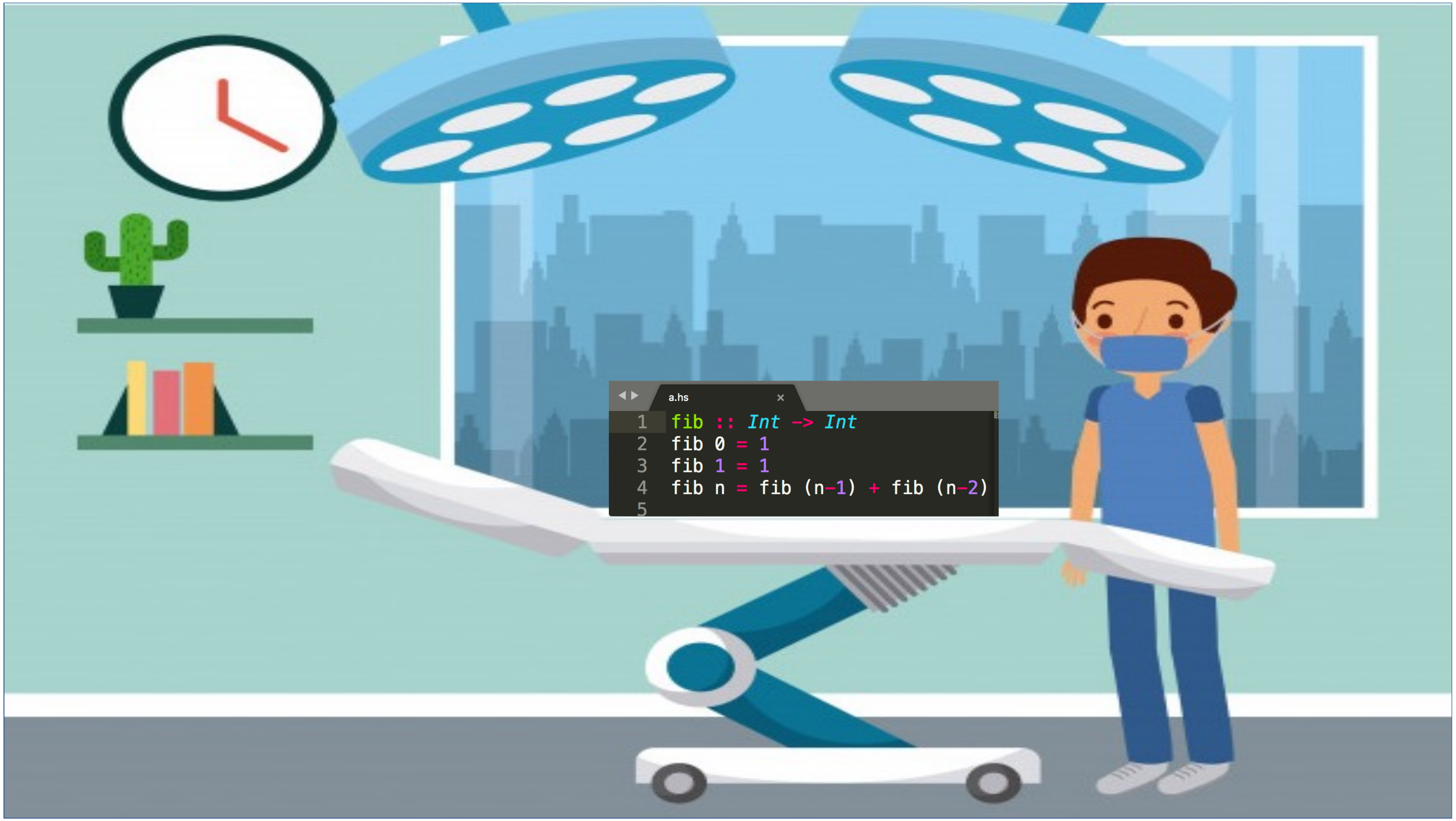
`DIVIDE & CONQUER'



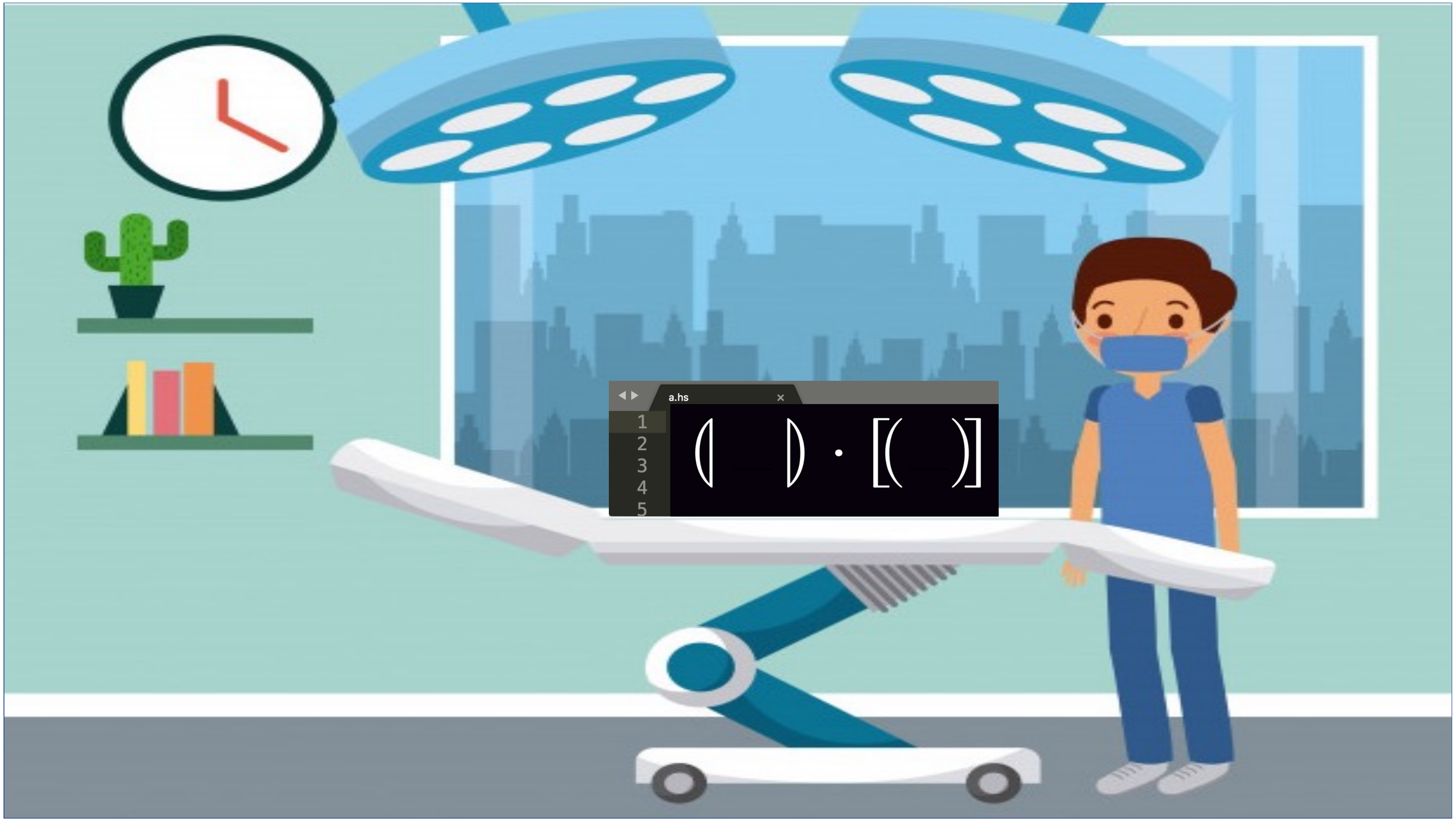
$$C \xleftarrow{(\!|\,\_\,|\!)} \mathsf{T} \xleftarrow{[\!(\_)\!]} A$$

Cálculo de Programas
Aula T10(a)

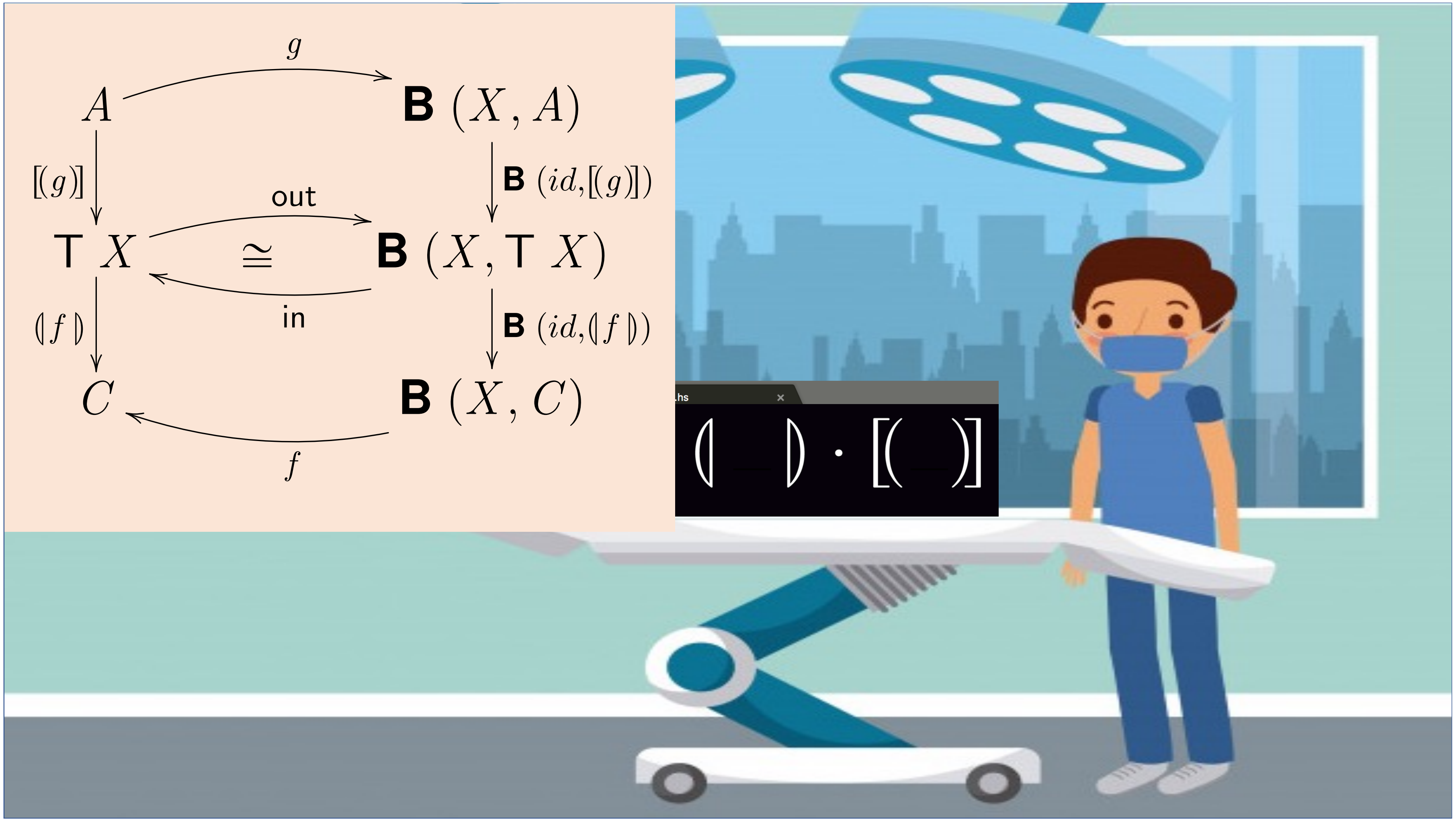




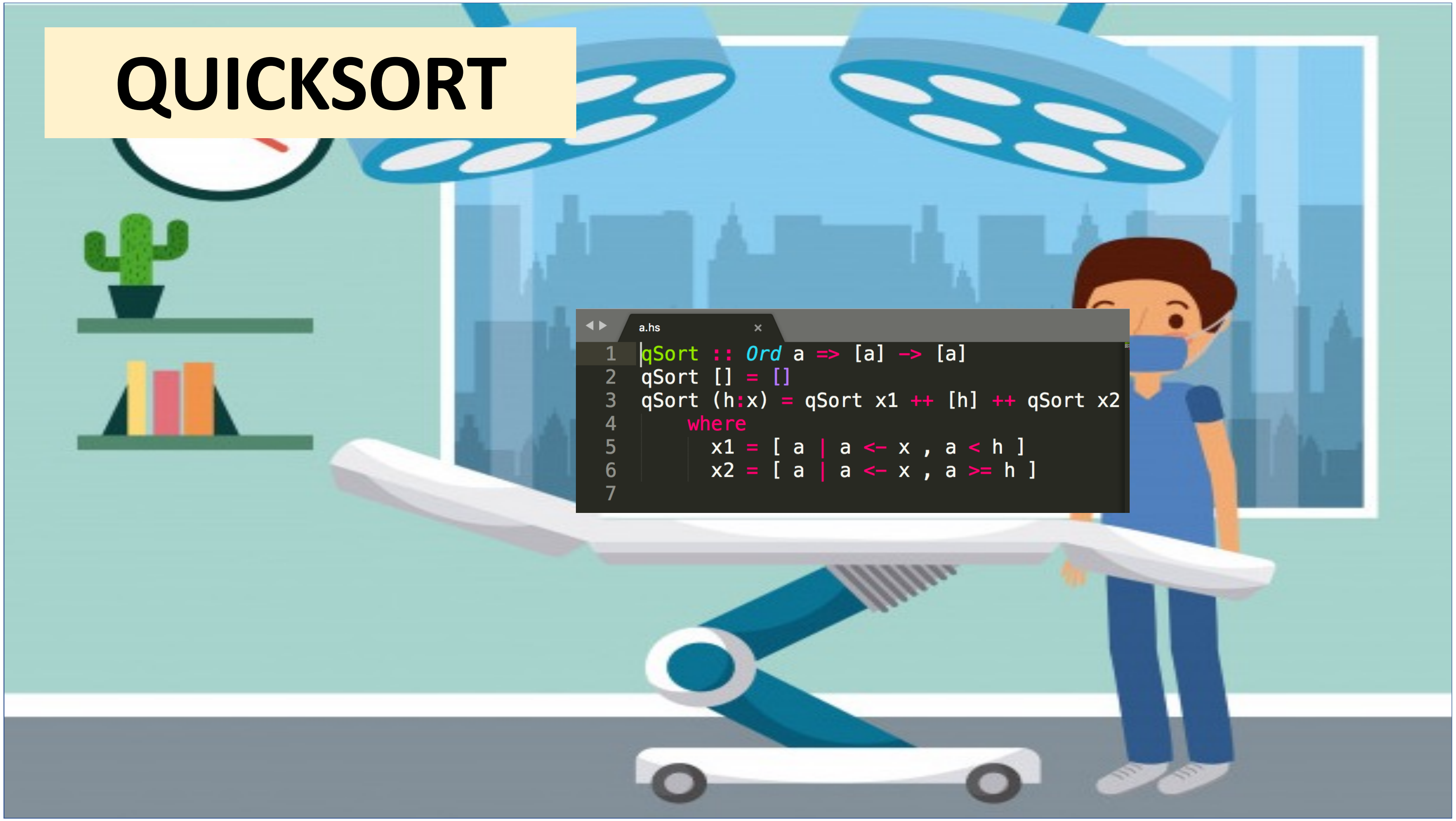
a.hs
fib :: Int -> Int
fib 0 = 1
fib 1 = 1
fib n = fib (n-1) + fib (n-2)



a.hs

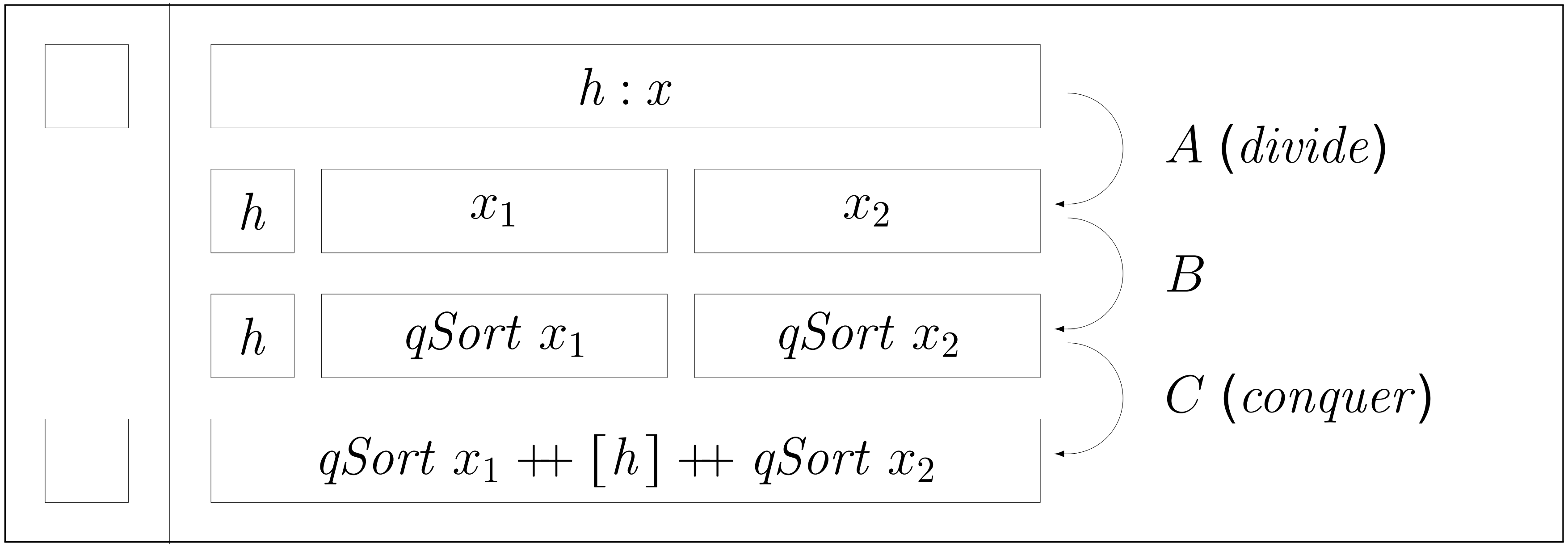


g
A
B (X, A)
[(g)]
B (id,[(g)])
out
T X
≅
B (X, T X)
in
(| f |)
B (id,(| f |))
C
B (X, C)
f
.hs
(| |) · [( )]


QUICKSORT
a.hs
qSort :: Ord a => [a] -> [a]
qSort [] = []
qSort (h:x) = qSort x1 ++ [h] ++ qSort x2
where
x1 = [a | a <- x , a < h]
x2 = [a | a <- x , a >= h]

QUICKSORT

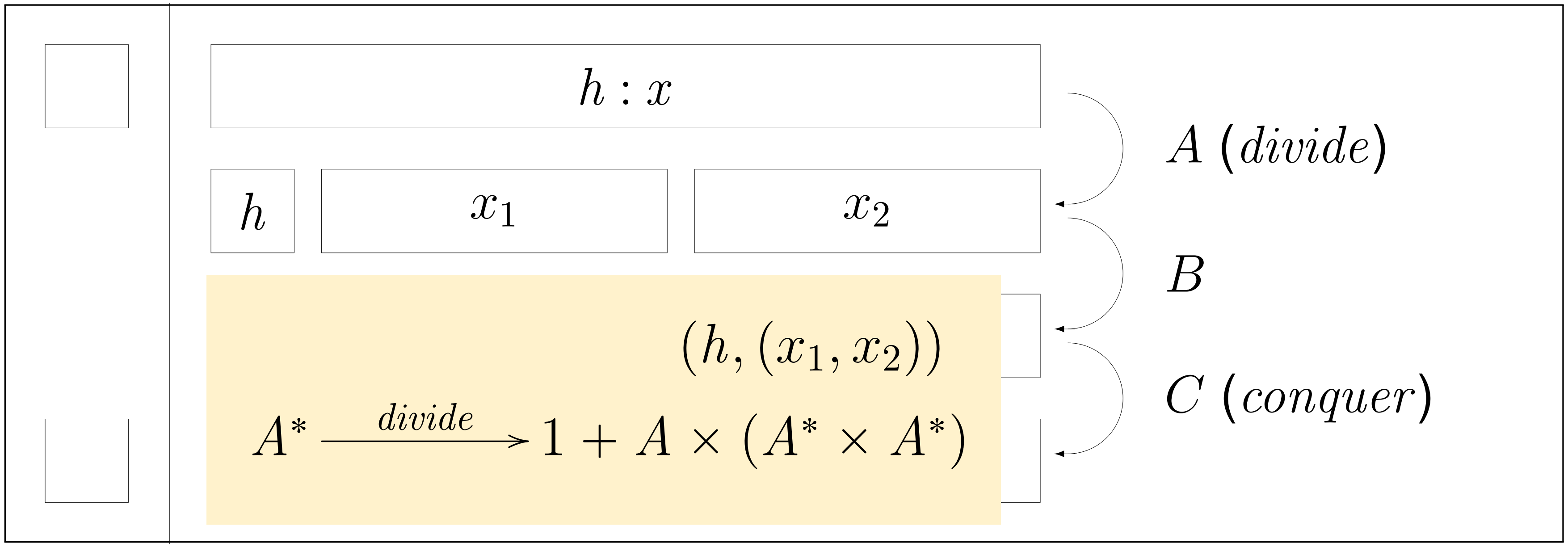
```haskell
qSort :: Ord a => [a] -> [a]
qSort [] = []
qSort (h:x) = qSort x1 ++ [h] ++ qSort x2
    where
        x1 = [ a | a <- x , a < h ]
        x2 = [ a | a <- x , a >= h ]
```



QUICKSORT

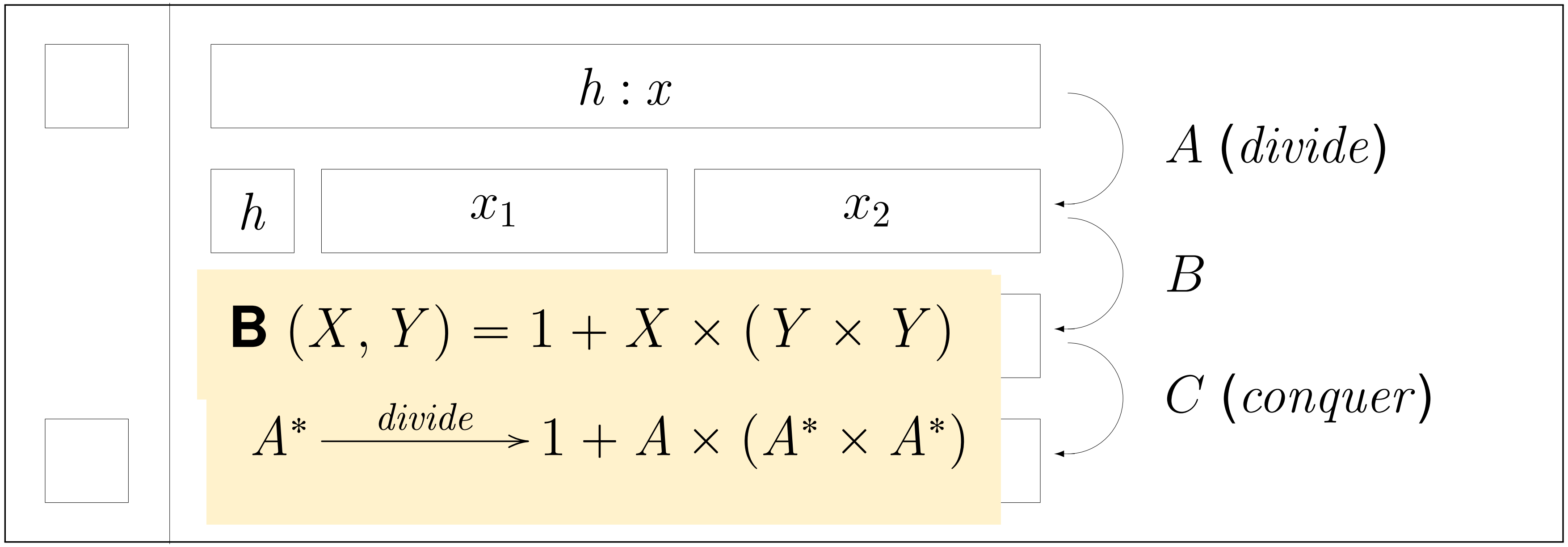
a.hs

```haskell
qSort :: Ord a => [a] -> [a]
qSort [] = []
qSort (h:x) = qSort x1 ++ [h] ++ qSort x2
    where
      x1 = [ a | a <- x , a < h ]
      x2 = [ a | a <- x , a >= h ]
```



QUICKSORT

```haskell
qSort :: Ord a => [a] -> [a]
qSort [] = []
qSort (h:x) = qSort x1 ++ [h] ++ qSort x2
    where
      x1 = [ a | a <- x , a < h ]
      x2 = [ a | a <- x , a >= h ]
```



$$\mathbf{B}\ (X, Y) = 1 + X \times (Y \times Y)$$

$$A^* \xrightarrow{divide} 1 + A \times (A^* \times A^*)$$

QUICKSORT

```
qSort :: Ord a => [a] -> [a]
qSort [] = []
qSort (h:x) = qSort x1 ++ [h] ++ qSort x2
    where
      x1 = [ a | a <- x , a < h ]
      x2 = [ a | a <- x , a >= h ]
```

$$\begin{cases} qSort \cdot \mathsf{nil} = \mathsf{nil} \\ qSort \cdot \mathsf{cons} = f_2 \cdot (id \times (qSort \times qSort)) \cdot g_2 \end{cases}$$

$$f_2\ (h, (y_1, y_2)) = y_1 +\!\!+ [h] +\!\!+ y_2$$

$$\begin{array}{l} g_2\ (h, x) = (h, (x_1, x_2)) \\ \quad \textbf{where} \\ \qquad x_1 = [a \mid a \leftarrow x, a < h] \\ \qquad x_2 = [a \mid a \leftarrow x, a \geqslant h] \end{array}$$

QUICKSORT

$$\mathsf{B}\,(X, Y) = 1 + X \times (Y \times Y)$$

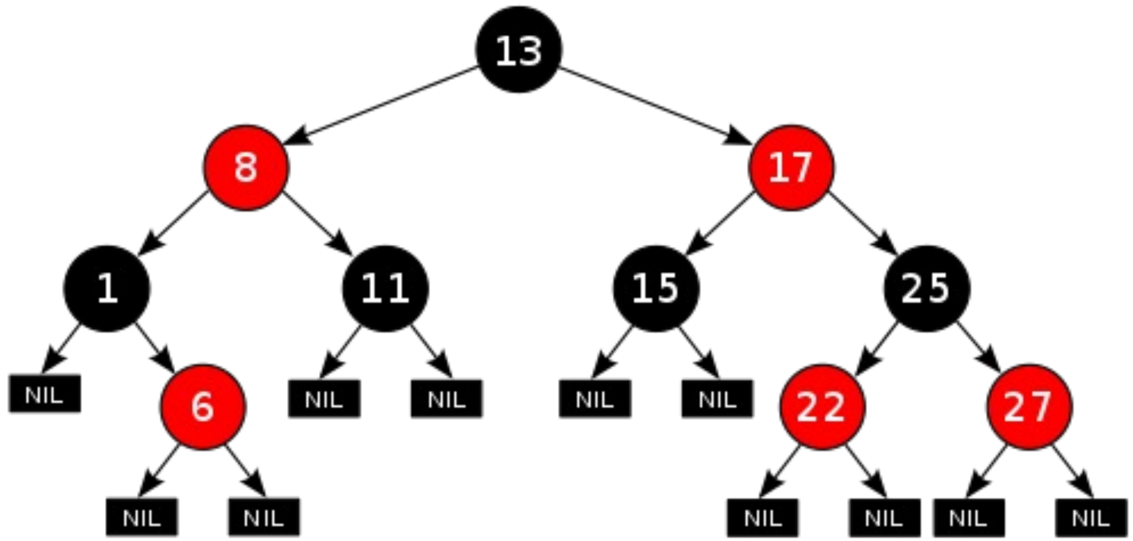
$$\begin{cases} qSort \cdot \mathsf{nil} = \mathsf{nil} \\ qSort \cdot \mathsf{cons} = f_2 \cdot (id \times (qSort \times qSort)) \cdot g_2 \end{cases}$$

$\equiv$ { fusão-$+$, absorção-$+$, eq-$+$ etc }

$$qSort \cdot \mathsf{in} = [\mathsf{nil}, f_2] \cdot (id + id \times qSort^2) \cdot (id + g_2)$$

$\equiv$ { isomorfismo in / out }

$$qSort = \underbrace{[\mathsf{nil}, f_2]}_{conquer} \cdot (\underbrace{id + id \times qSort^2}_{\mathsf{B}\,(id, qSort)} \cdot \underbrace{(id + g_2) \cdot \mathsf{out}}_{divide}$$




```
data BTree a = Empty | Node (a, (BTree a, BTree a))
```

QUICKSORT

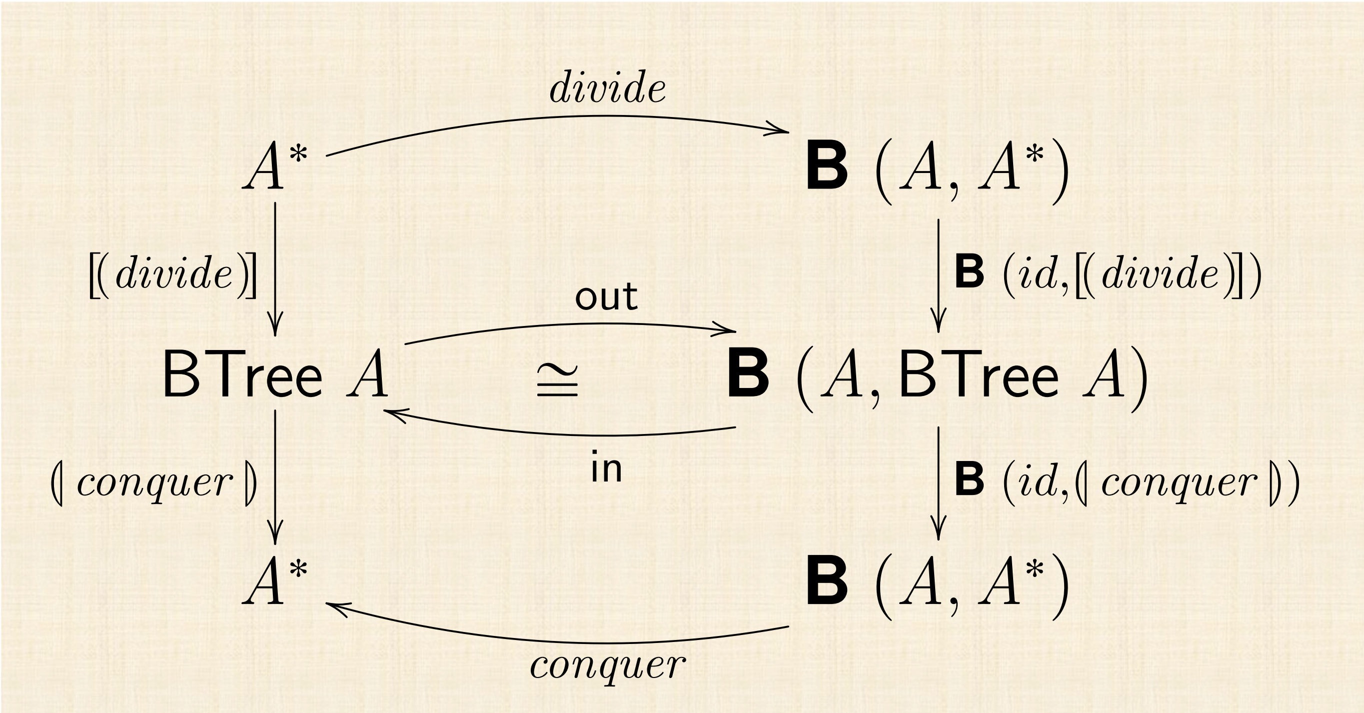
$$\mathsf{B}\,(X, Y) = 1 + X \times (Y \times Y)$$

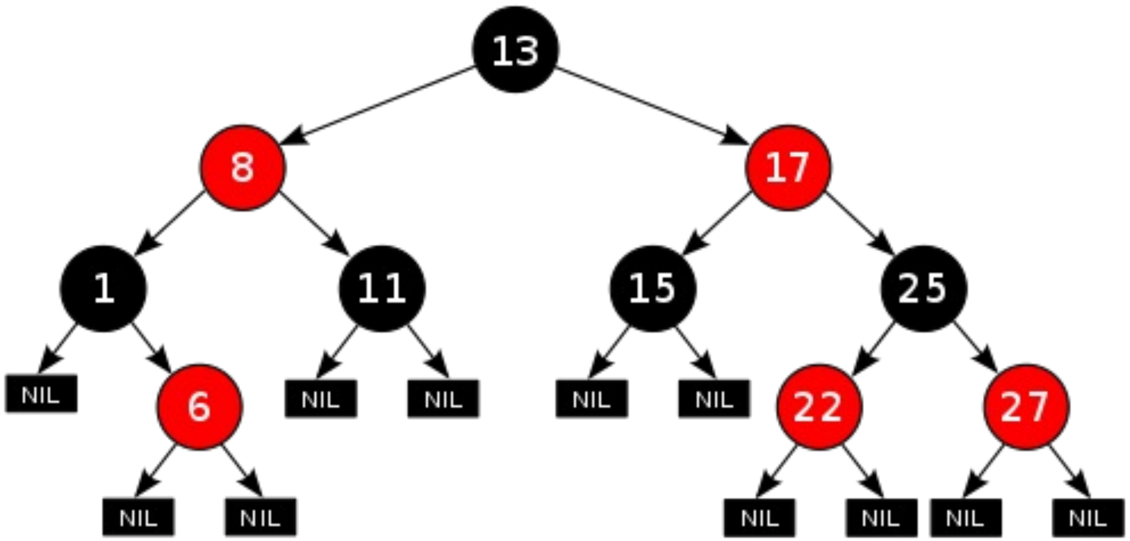
Description	T X	B (X, Y)	B (id, f)	B (f, id)
"Right" Lists	List X	$1 + X \times Y$	$id + id \times f$	$id + f \times id$
"Left" Lists	LList X	$1 + Y \times X$	$id + f \times id$	$id + id \times f$
Non-empty Lists	NList X	[illegible] $+ X \times Y$	$id + id \times f$	$f + f \times id$
Binary Trees	BTree X	$1 + X \times Y^2$	$id + id \times f^2$	$id + f \times id$
"Leaf" Trees	LTree X	$X + Y^2$	$id + f^2$	$f + id$

QUICKSORT

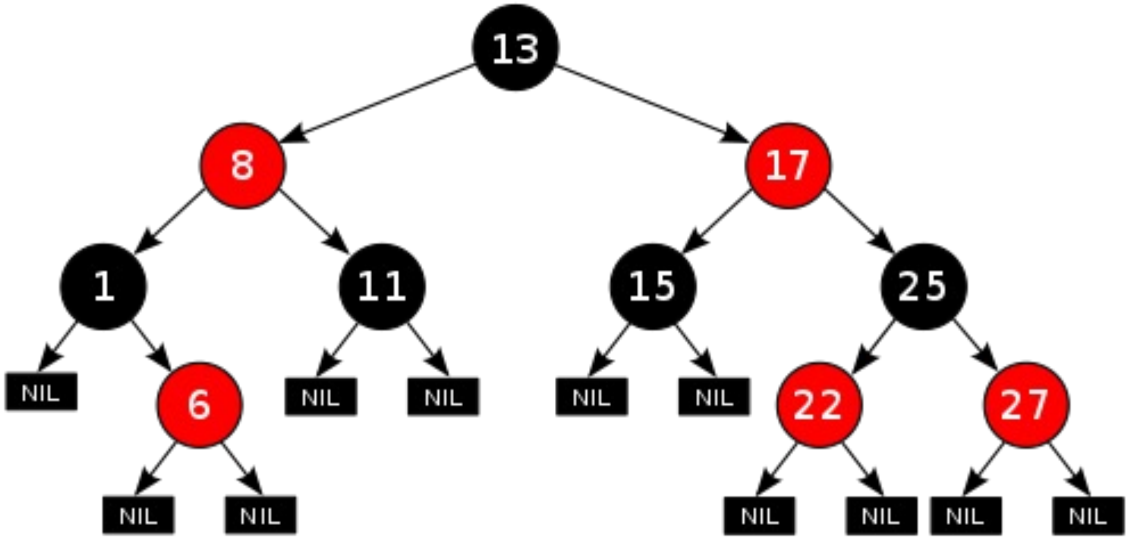
a.hs

```haskell
qSort :: Ord a => [a] -> [a]
qSort [] = []
qSort (h:x) = qSort x1 ++ [h] ++ qSort x2
    where
      x1 = [ a | a <- x , a < h ]
      x2 = [ a | a <- x , a >= h ]
```

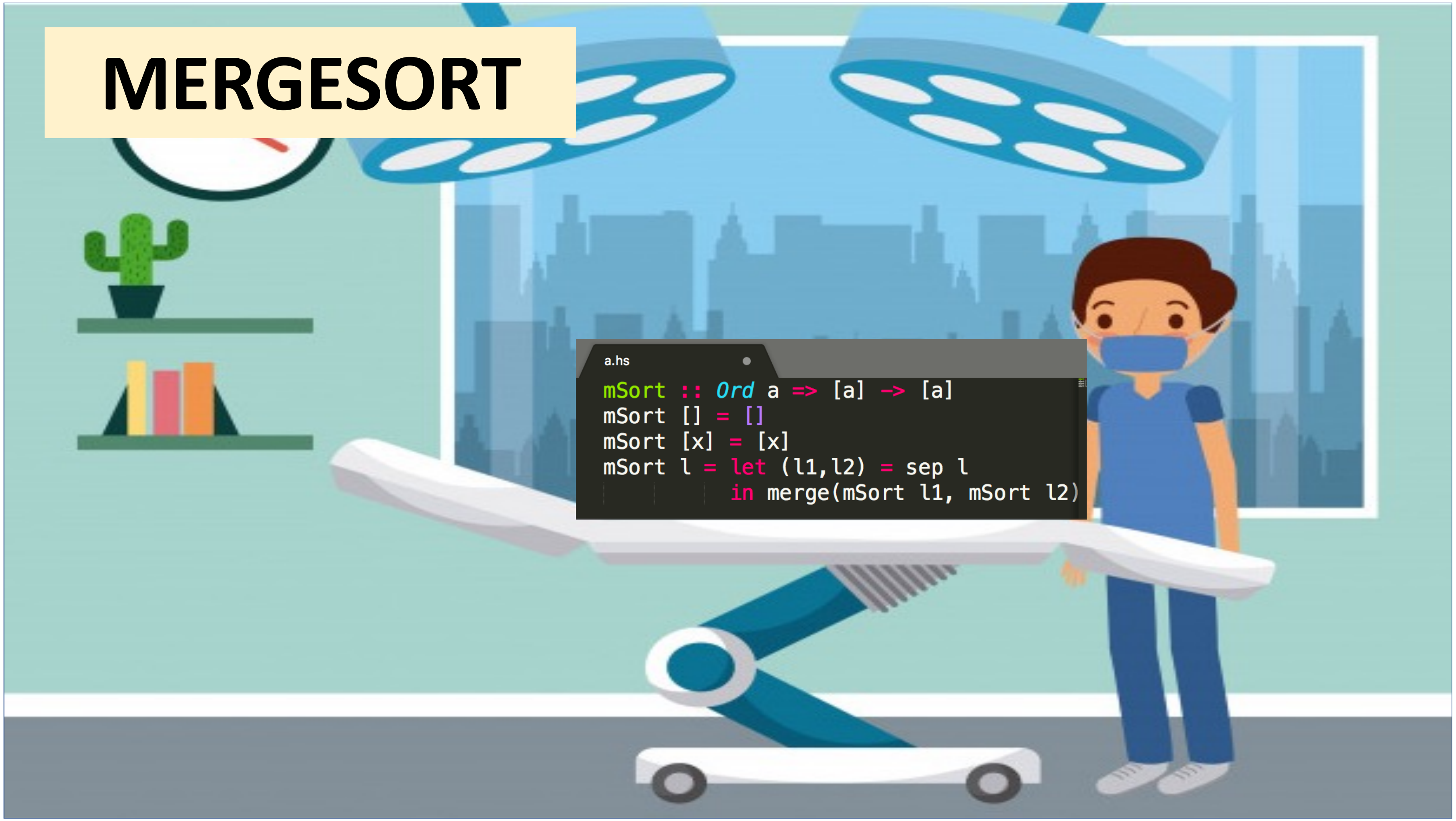





```haskell
t = anaB divide [13,8,17,1,6,11,25,15,27,22]

data BTree a = Empty | Node (a, (BTree a, BTree a))
```



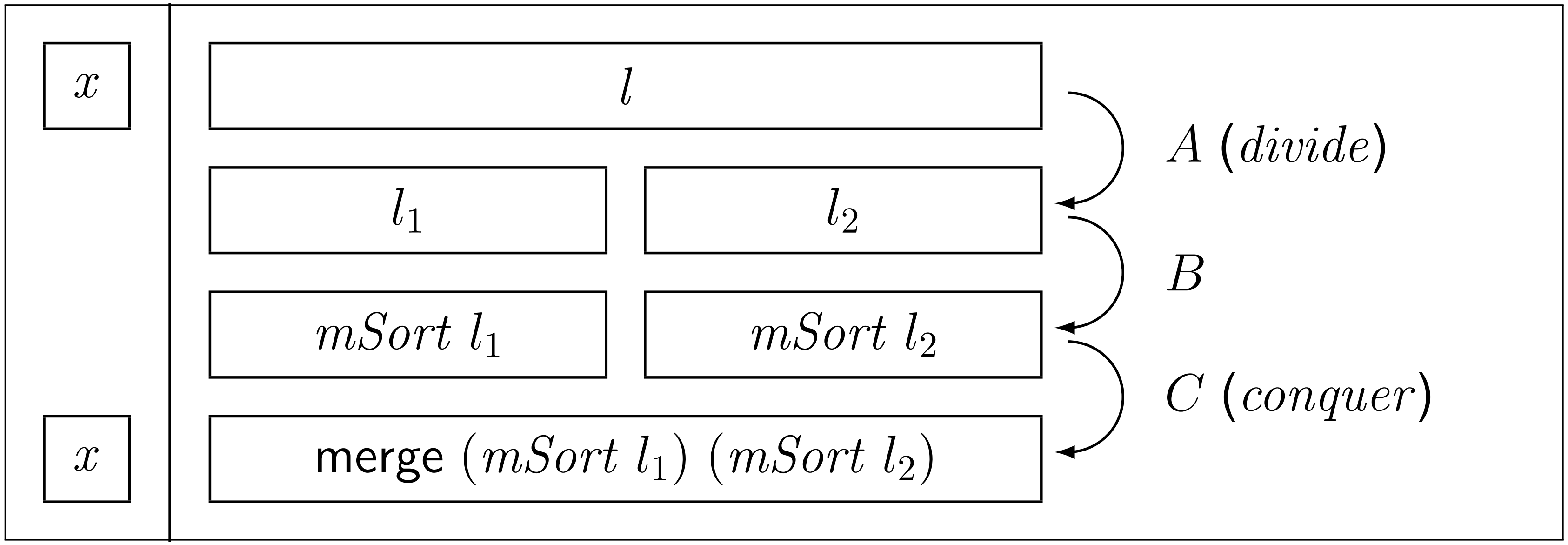

```
*Cp> anaB divide [13,8,17,1,6,11,25,15,27,22]
Node (13,(Node (8,(Node (1,(Empty,Node (6,(Empty,Empty)))),Node
(11,(Empty,Empty)))),Node (17,(Node (15,(Empty,Empty)),Node (25,
(Node (22,(Empty,Empty)),Node (27,(Empty,Empty))))))))
```


MERGESORT
a.hs
mSort :: Ord a => [a] -> [a]
mSort [] = []
mSort [x] = [x]
mSort l = let (l1,l2) = sep l
in merge(mSort l1, mSort l2)

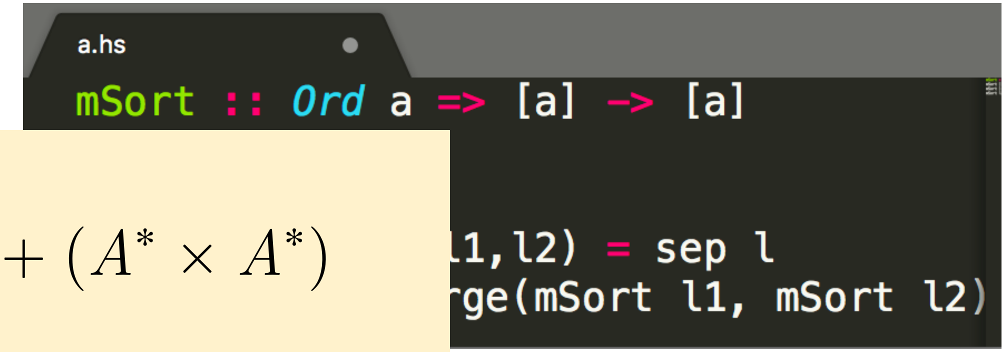
MERGESORT

a.hs

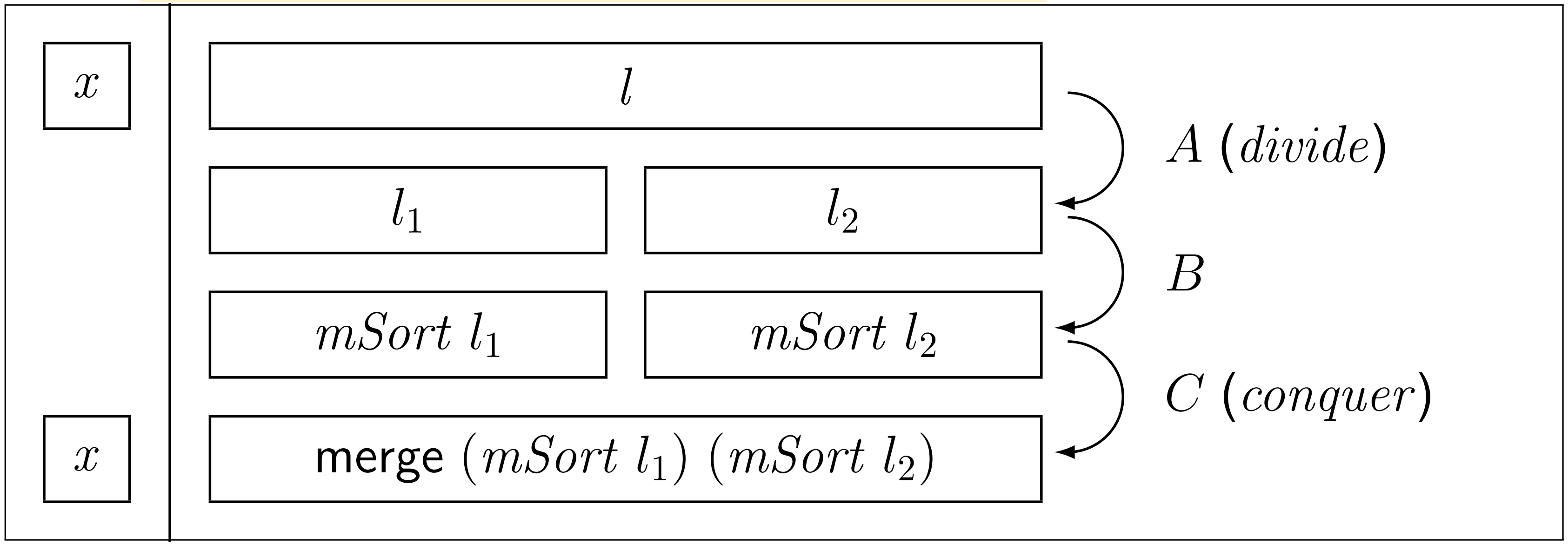
```haskell
mSort :: Ord a => [a] -> [a]
mSort [] = []
mSort [x] = [x]
mSort l = let (l1,l2) = sep l
          in merge(mSort l1, mSort l2)
```



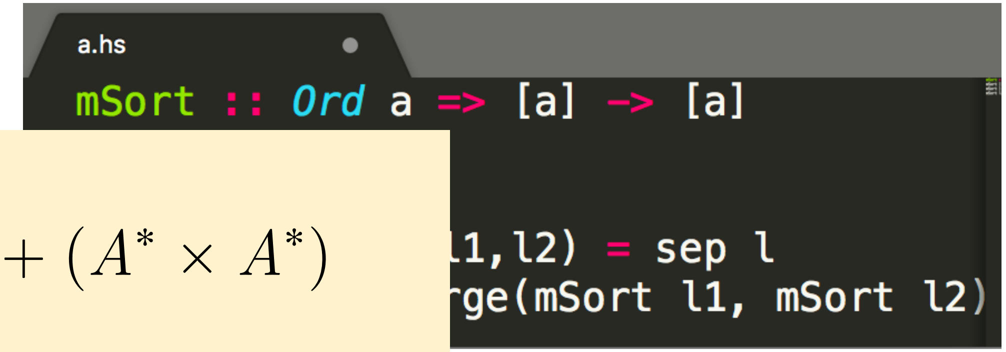
MERGESORT



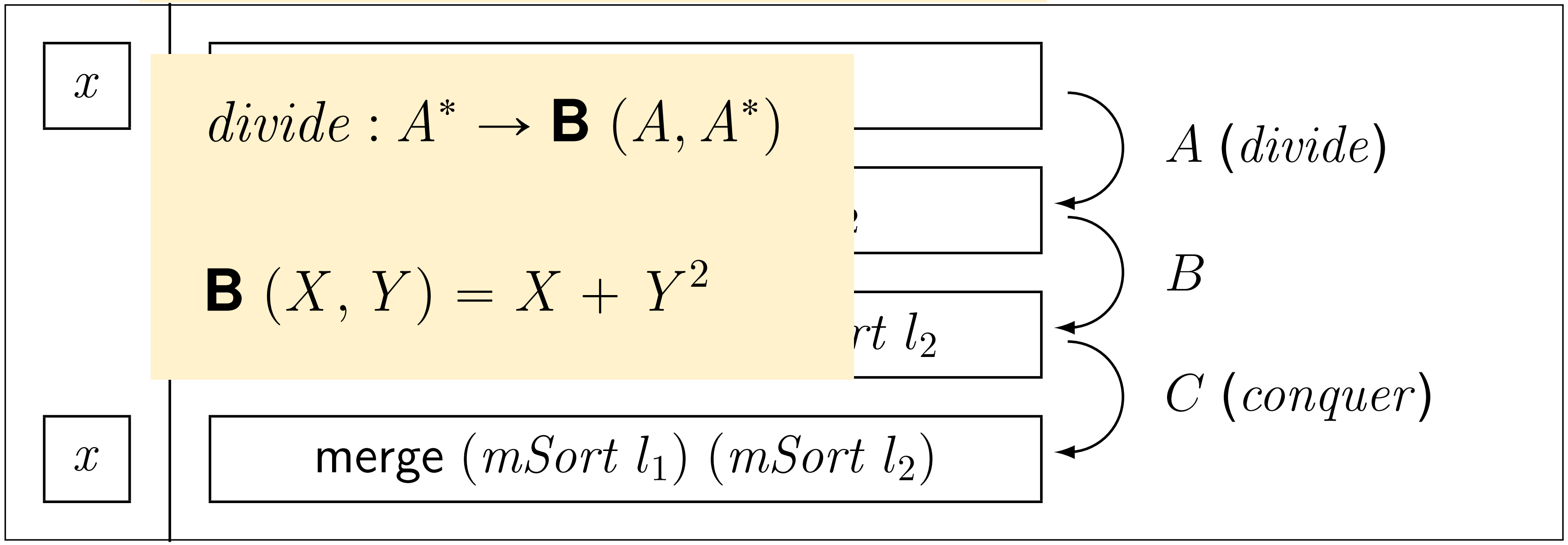
$$divide : A^* \to A + (A^* \times A^*)$$



MERGESORT



$$divide : A^* \rightarrow A + (A^* \times A^*)$$



$$divide : A^* \rightarrow \mathbf{B}\,(A, A^*)$$

$$\mathbf{B}\,(X, Y) = X + Y^2$$

MERGESORT

```
a.hs
mSort :: Ord a => [a] -> [a]

l1,l2) = sep l
rge(mSort l1, mSort l2)
```

$$divide : A^* \to A + (A^* \times A^*)$$

x

merge $(mSort\ l_1)\ (mSort\ l_2)$

x

$A\ (divide)$

B

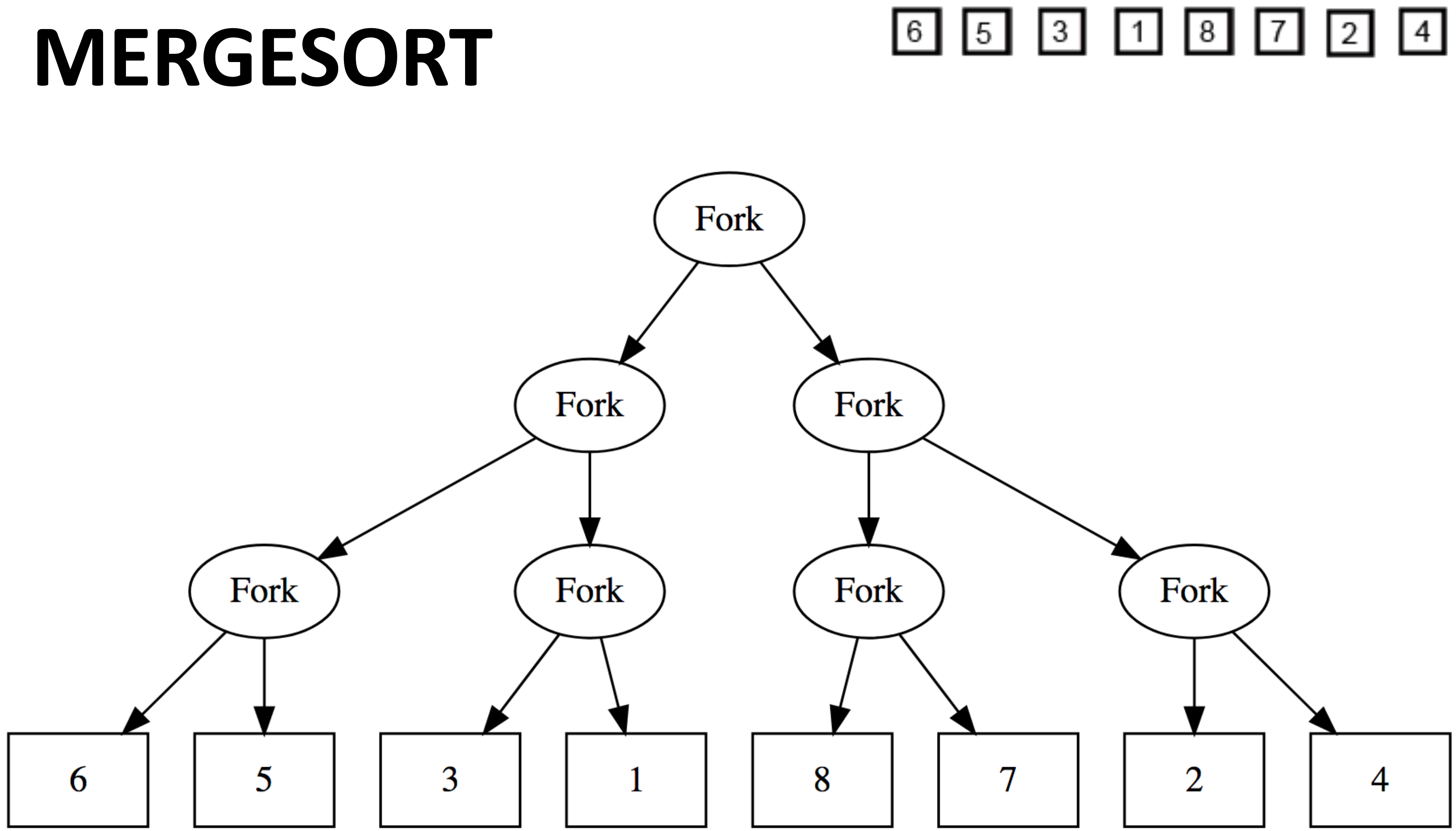
$uer)$

$$divide : A^* \to \mathbf{B}\,(A, A^*)$$

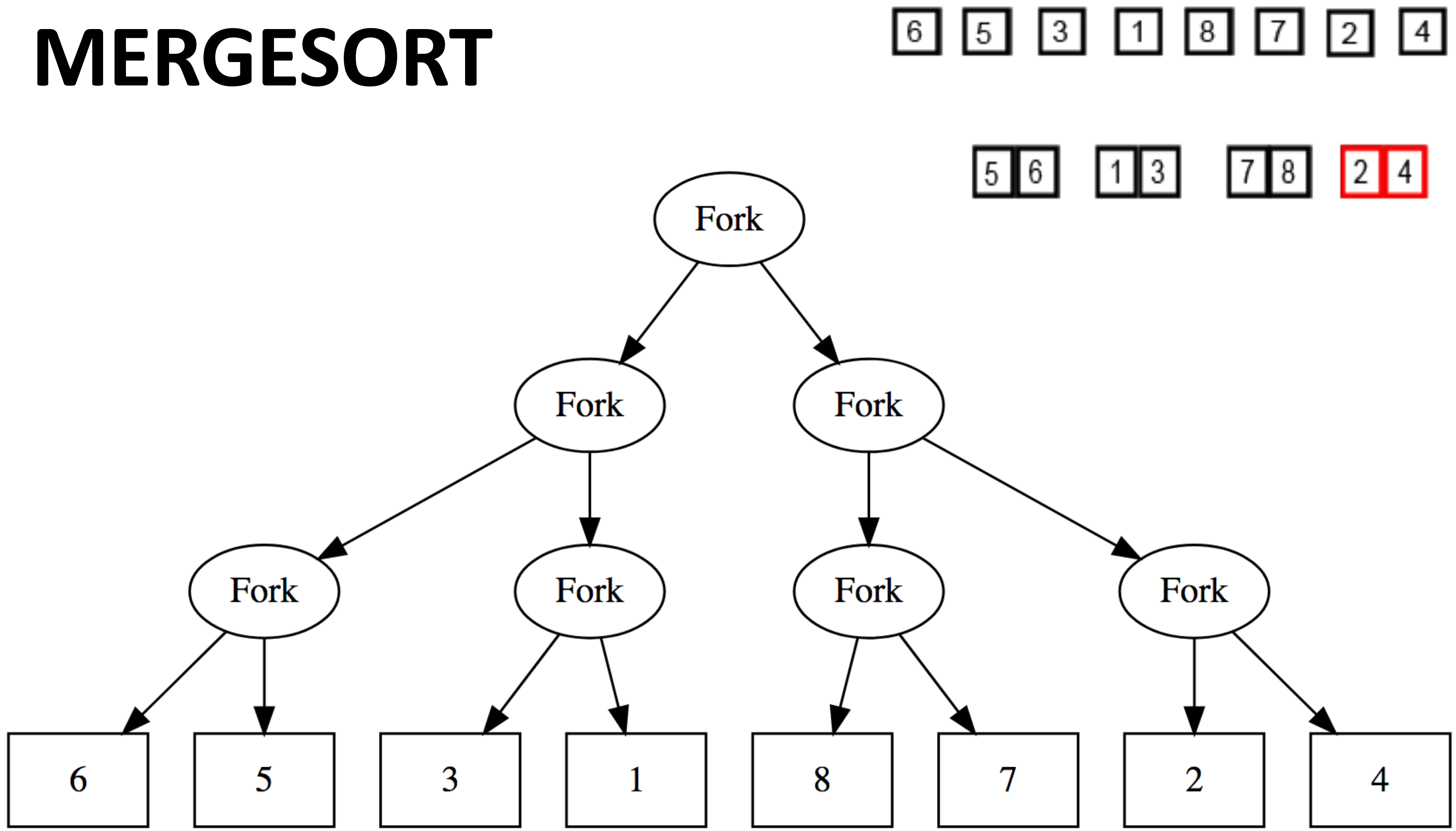
$$\mathbf{B}\,(X, Y) = X + Y^2$$

"Leaf" Trees	LTree X	$X + Y^2$

MERGESORT



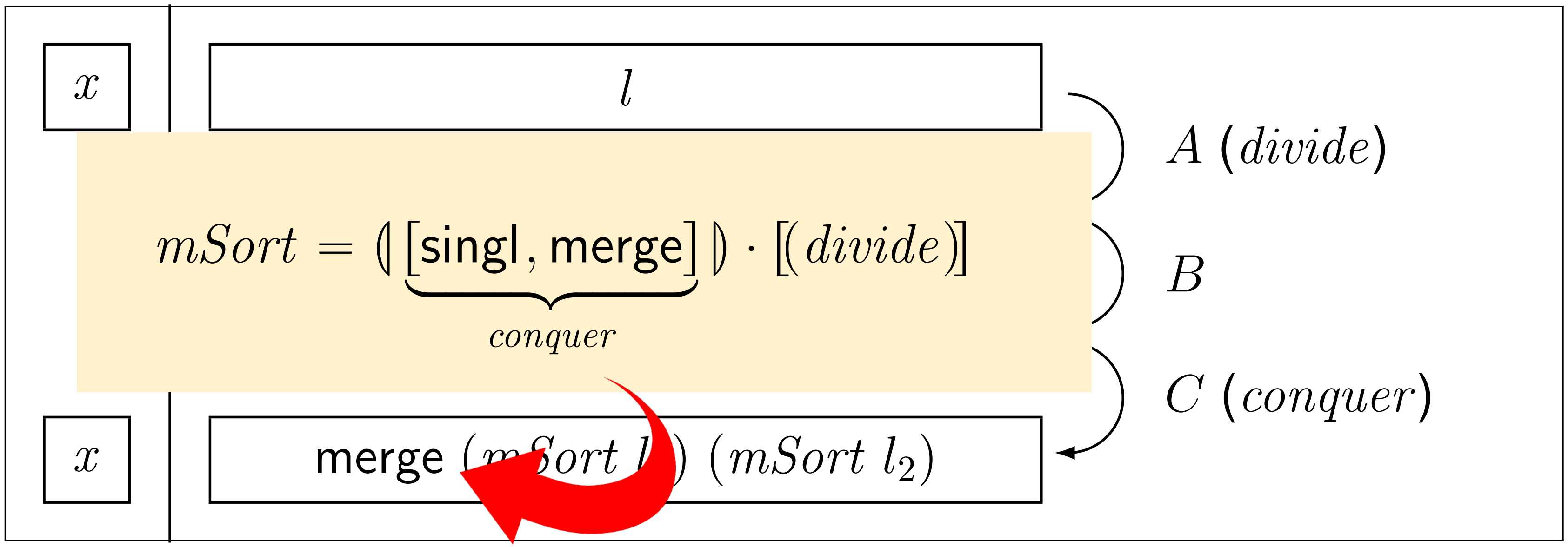
MERGESORT



6 5 3 1 8 7 2 4
5 6 1 3 7 8 2 4
Fork
Fork
Fork
Fork
Fork
Fork
Fork
6
5
3
1
8
7
2
4

MERGESORT

```haskell
a.hs
mSort :: Ord a => [a] -> [a]
mSort [] = []
mSort [x] = [x]
mSort l = let (l1,l2) = sep l
              in merge(mSort l1, mSort l2)
```

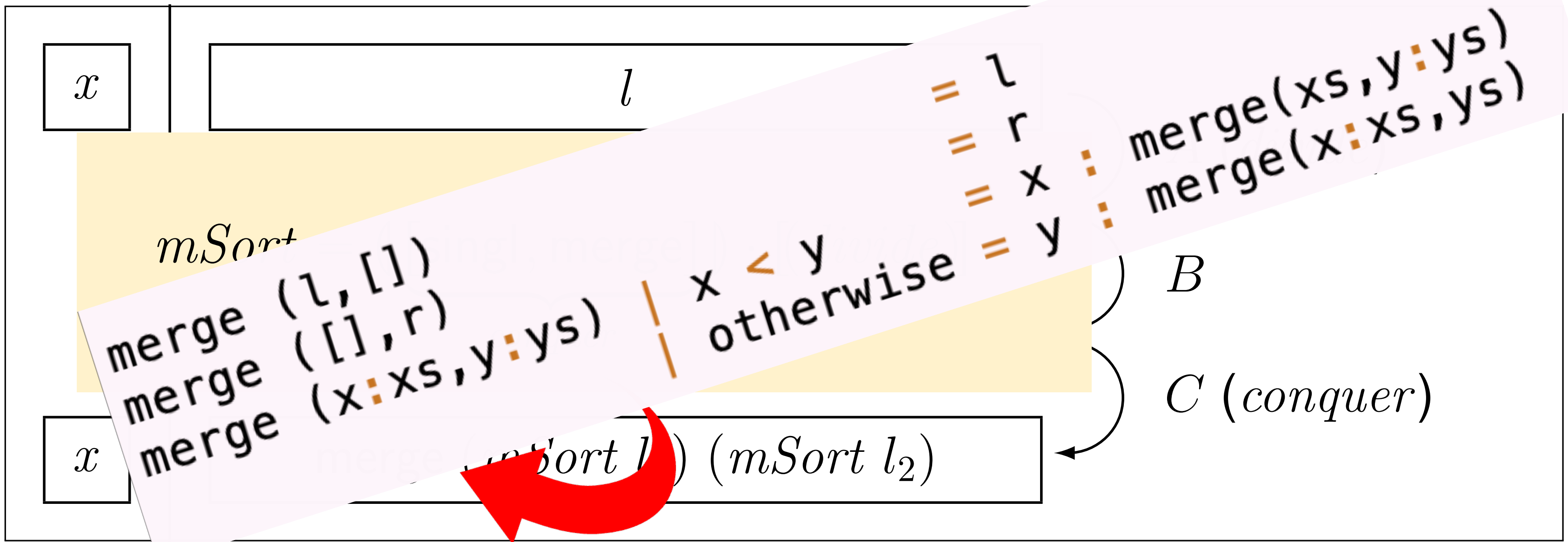


MERGESORT

a.hs

```haskell
mSort :: Ord a => [a] -> [a]
mSort [] = []
mSort [x] = [x]
mSort l = let (l1,l2) = sep l
          in merge(mSort l1, mSort l2)
```

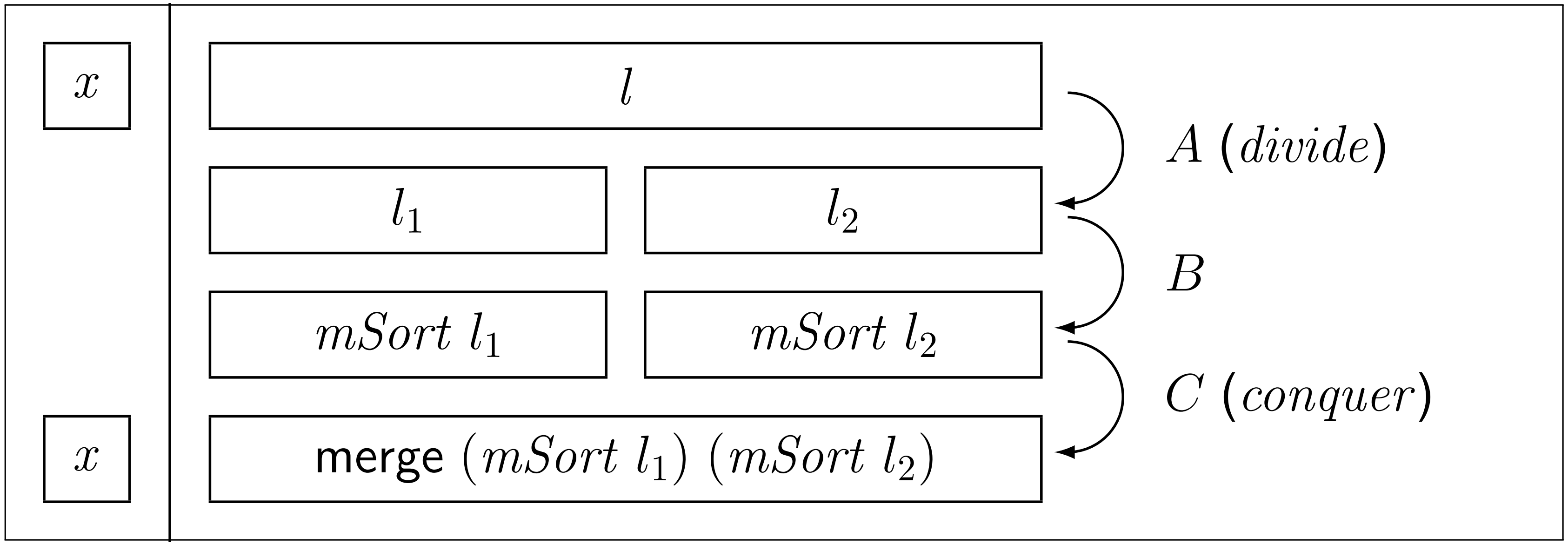
```haskell
merge (l,[])                    = l
merge ([],r)                    = r
merge (x:xs,y:ys) | x < y       = x : merge(xs,y:ys)
                  | otherwise   = y : merge(x:xs,ys)
```



MERGESORT

a.hs

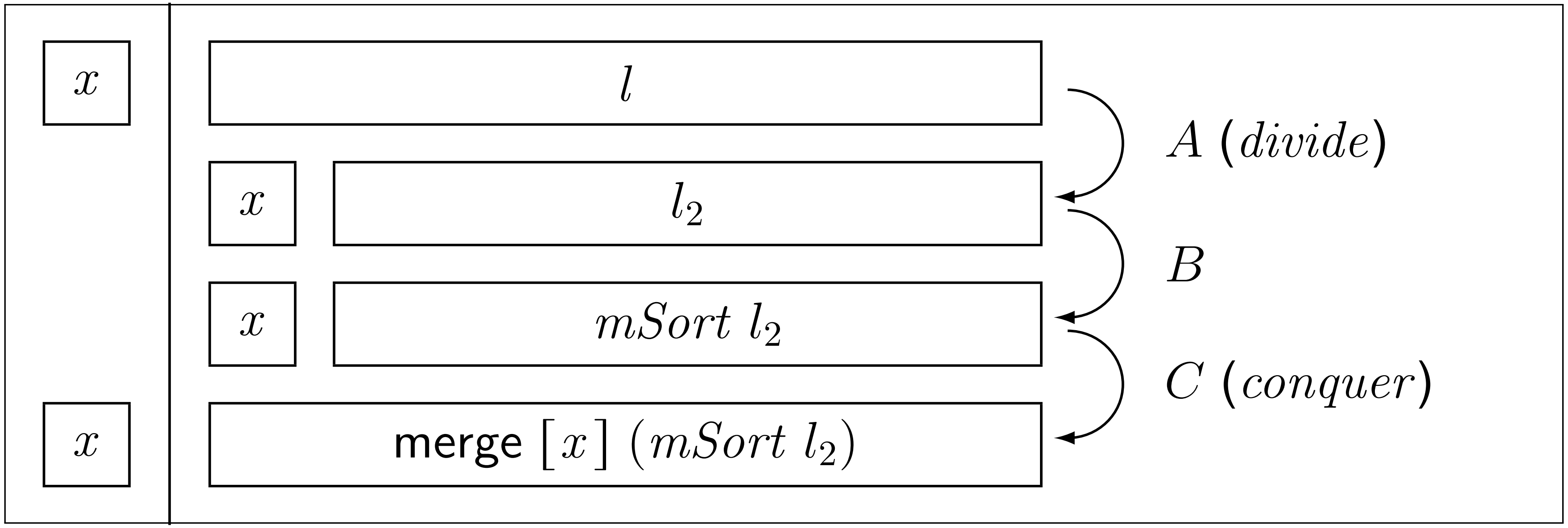
```haskell
mSort :: Ord a => [a] -> [a]
mSort [] = []
mSort [x] = [x]
mSort l = let (l1,l2) = sep l
          in merge(mSort l1, mSort l2)
```



MERGESORT

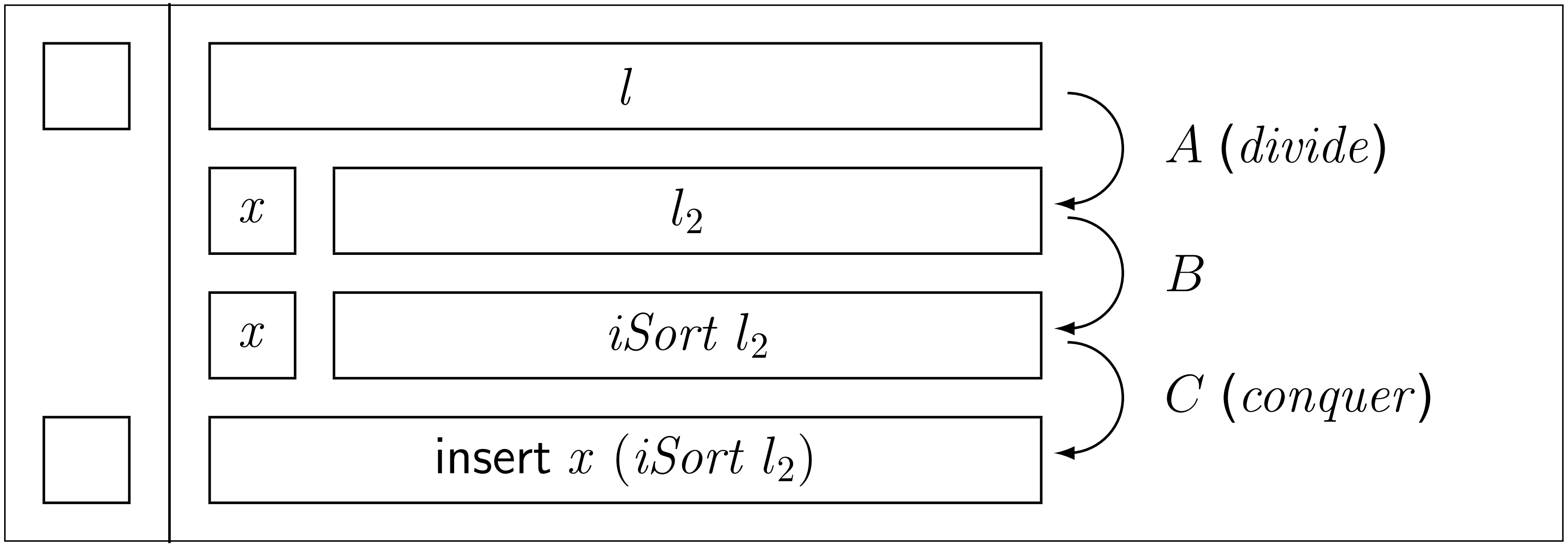
Particular case:

```
merge :: Ord a => ([a], [a]) -> [a]
merge ([x],[])                   = [x]
merge ([],r)                     = r
merge ([x],y:ys) | x < y         = x : merge([],y:ys)
                 ) | otherwise = y : merge(x:[],ys
```

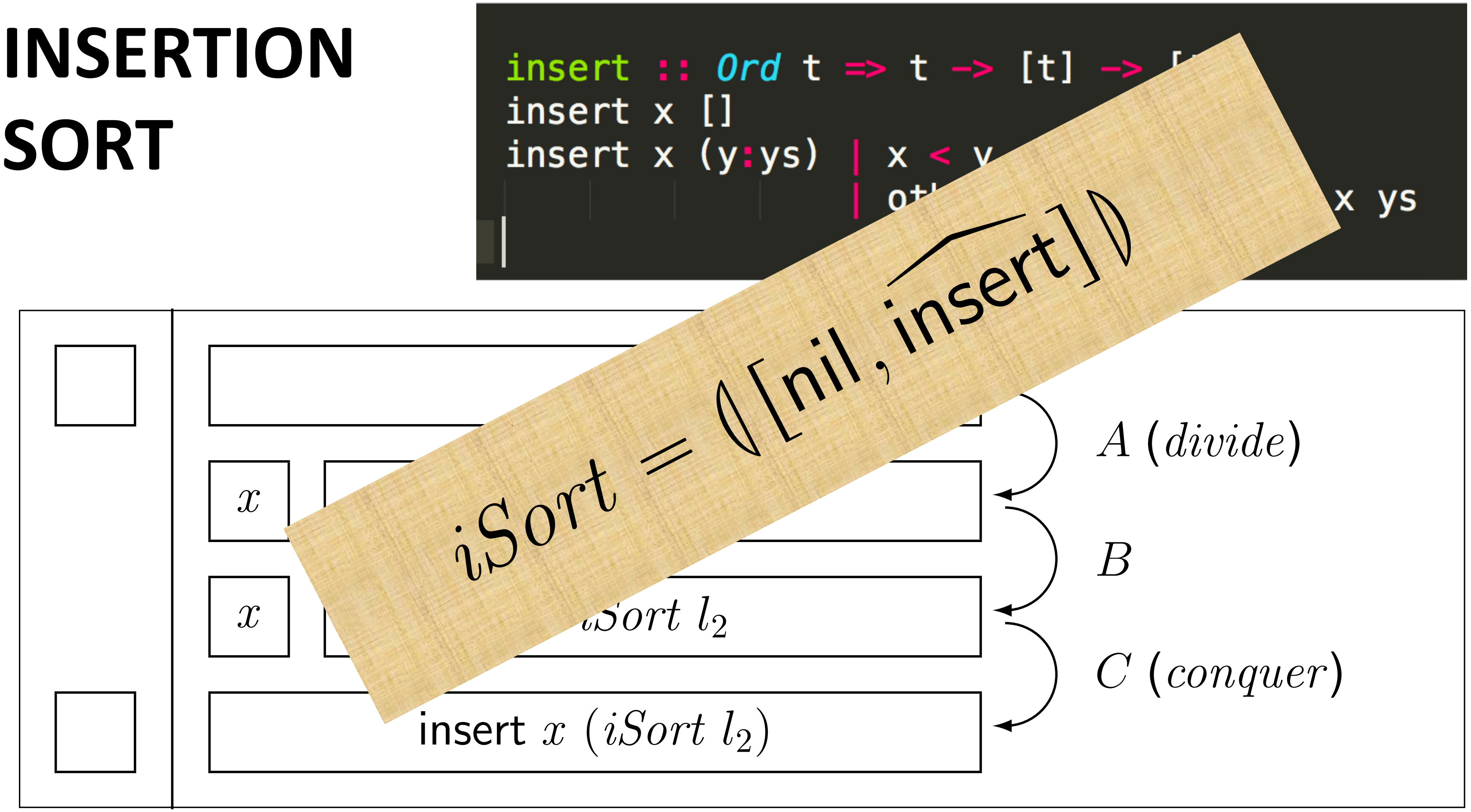


INSERTION SORT

```
insert :: Ord t => t -> [t] -> [t]
insert x []                    = [x]
insert x (y:ys) | x < y        = x:y:ys
                | otherwise    = y:insert x ys
```



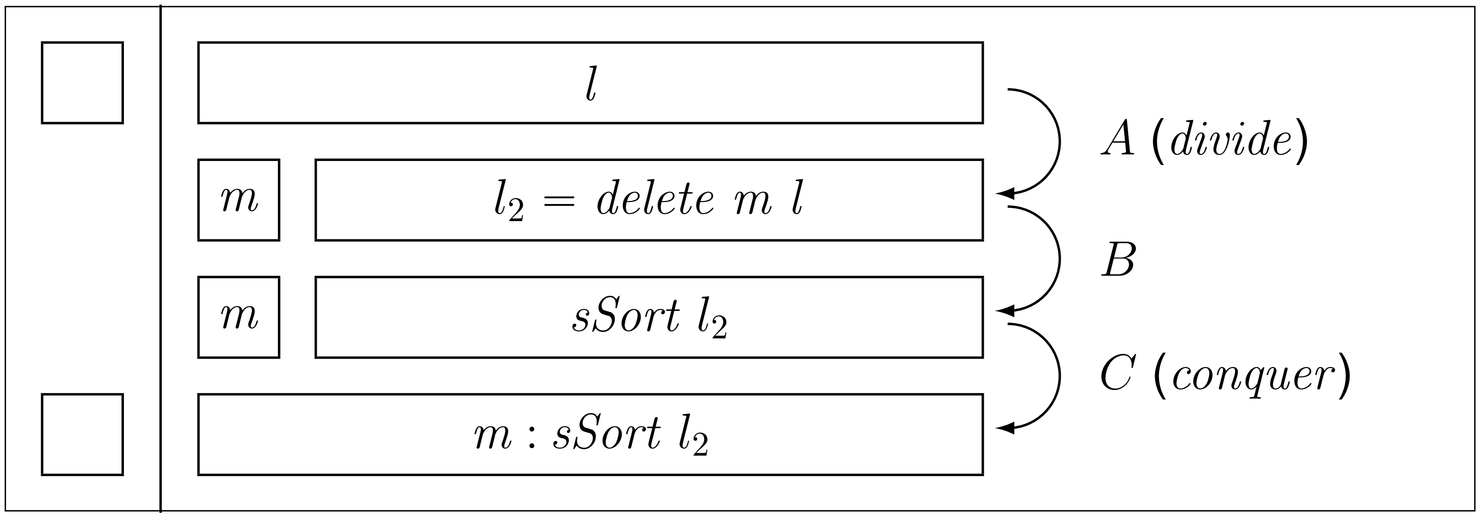
INSERTION SORT



SELECTION SORT

```haskell

sSort :: Ord a => [a] -> [a]
sSort = anaL divide where
   divide [] = i1()
   divide (xs) = let m = minimum xs
                 in  i2(m, delete m xs)
```

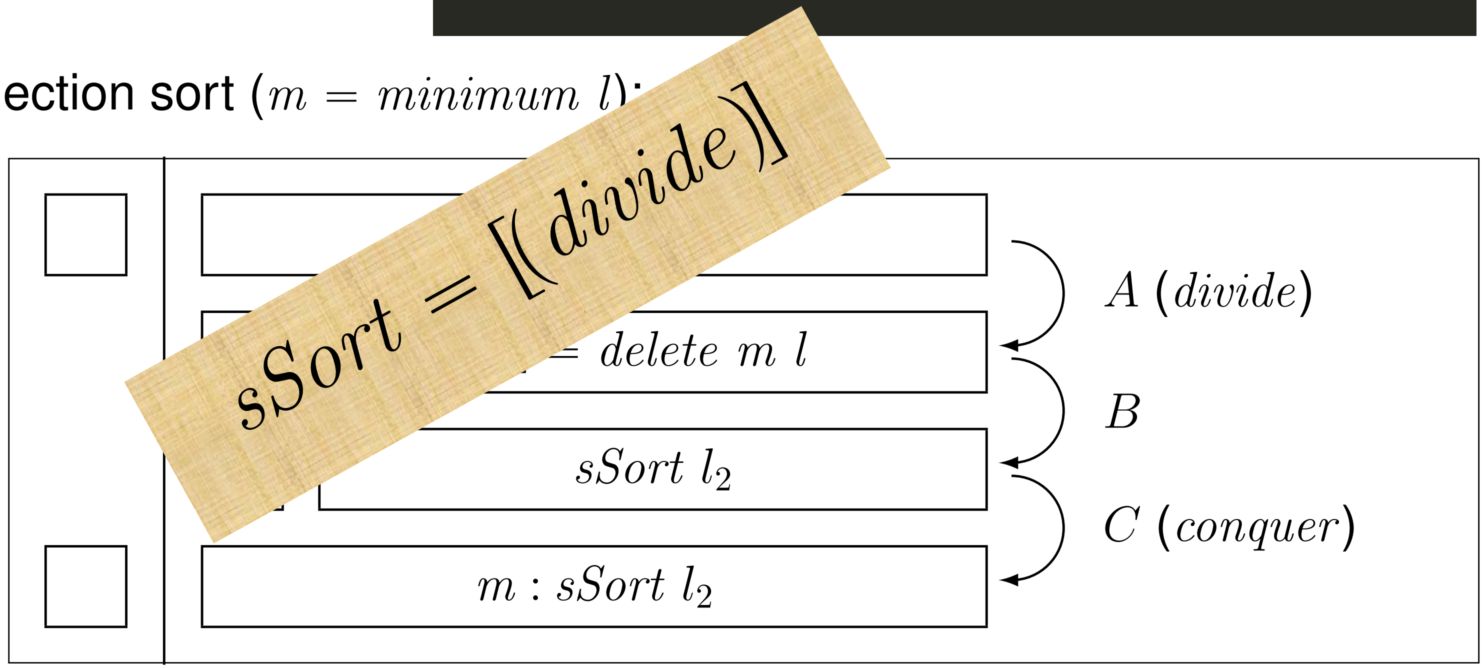
Selection sort ($m = minimum\ l$):



SELECTION SORT

```haskell

sSort :: Ord a => [a] -> [a]
sSort = anaL divide where
   divide [] = i1()
   divide (xs) = let m = minimum xs
                 in  i2(m, delete m xs)
```

Selection sort ($m = minimum\ l$):



ANA, CATA & HILO

$$C \xleftarrow{(\!| f |\!)} \mathsf{T} \xleftarrow{[\!(g)\!]} A$$

$$[\![f , g]\!] = (\!| f |\!) \cdot [\!(g)\!]$$

ANA, CATA & HILO

$$[\![\mathsf{in}, g]\!] = [\!(g)\!]$$

$$[\![f, \mathsf{out}]\!] = (\!| f |\!)$$

Reflexion laws:

$$(\!| \mathsf{in} |\!) = id$$

$$[\!(\mathsf{out})\!] = id$$

$$C \xleftarrow{(\!| f |\!)} \mathsf{T} \xleftarrow{[\!(g)\!]} A$$

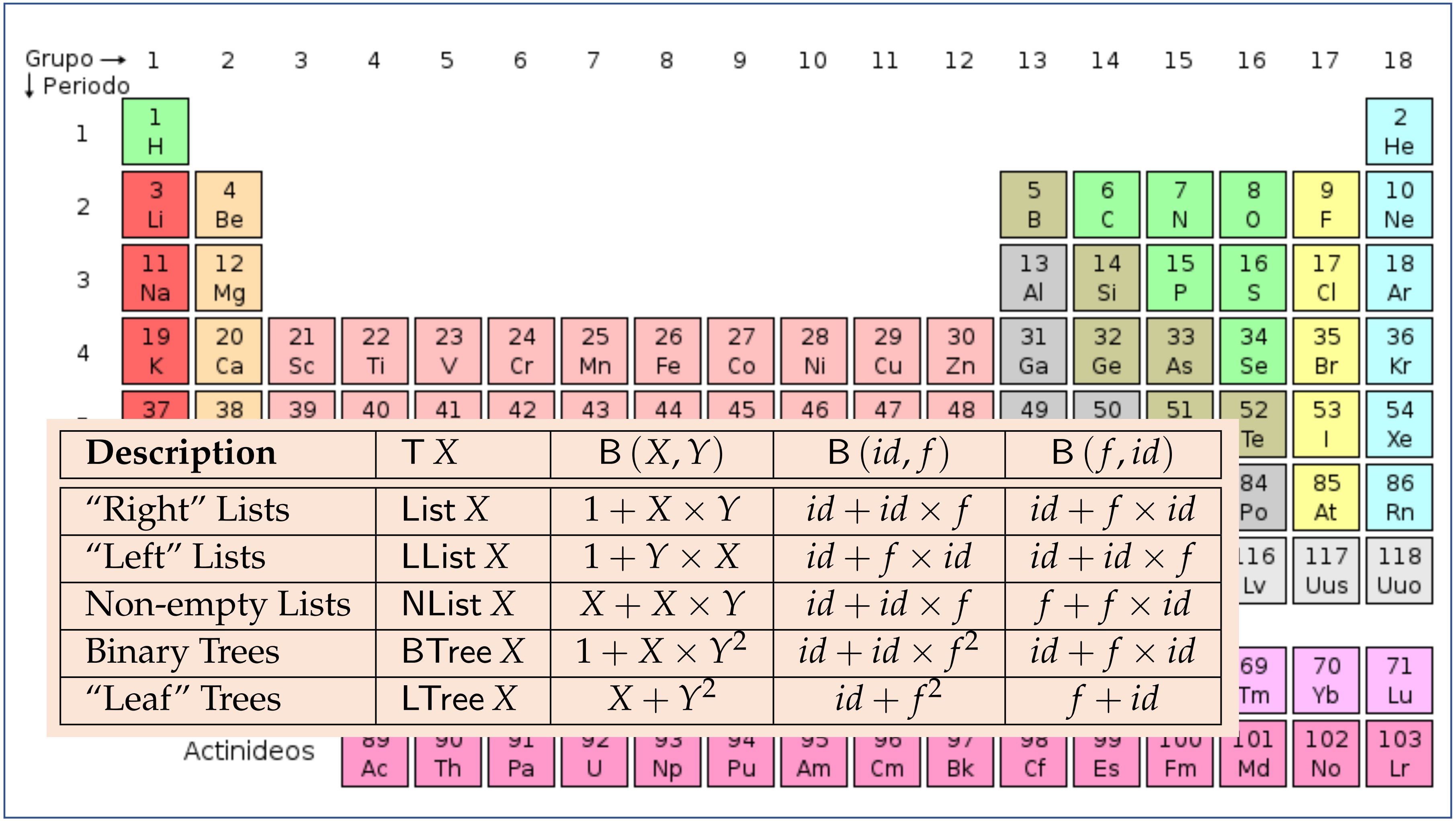
$$[\![f, g]\!] = (\!| f |\!) \cdot [\!(g)\!]$$

CLASSIFICAÇÃO

	Singleton	Equal-size
Easy Split/Hard Join	Insertion Sort	Merge Sort
Hard Split/Easy Join	Selection Sort	Quick Sort

NB:
'Split' = *divide*
'Join' = *conquer*

$$C \xleftarrow{(\!|\_|\!)} \mathsf{T} \xleftarrow{[\!(\_)\!]} A$$



Description	T X	B (X, Y)	B (id, f)	B (f, id)
"Right" Lists	List X	$1 + X \times Y$	$id + id \times f$	$id + f \times id$
"Left" Lists	LList X	$1 + Y \times X$	$id + f \times id$	$id + id \times f$
Non-empty Lists	NList X	$X + X \times Y$	$id + id \times f$	$f + f \times id$
Binary Trees	BTree X	$1 + X \times Y^2$	$id + id \times f^2$	$id + f \times id$
"Leaf" Trees	LTree X	$X + Y^2$	$id + f^2$	$f + id$

“Tabela periódica”

```haskell
qSort :: Ord a => [a] -> [a]
qSort [] = []
qSort (h:x) = qSort x1 ++ [h] ++ qSort x2
    where
      x1 = [ a | a <- x , a < h ]
      x2 = [ a | a <- x , a >= h ]
```

		B (X, Y)	$1 + Y$	$X + Y$	$1 + X \times Y$	$Z + X \times Y$	$X + Y^2$	$1 + X \times Y^2$
A	C	T X	$\mathbb{N}_0$	XNat X	X^*	SList X Z	LTree X	BTree X
$\mathbb{N}_0$	$\mathbb{N}_0$	Factorial			fac		$dfac$	
$\mathbb{N}_0$	$\mathbb{N}_0$	Misc. em $\mathbb{N}_0$	$(n*), (n+), \_^n$ etc		sq		dsq, fib	
$\mathbb{N}_0$	$\mathbb{N}_0{}^*$	Séries			$odds$, $evens$			
$\mathbb{N}_0 \times X^*$	X^*	Selecção		$udrop$	$utake$			
$\mathbb{R}$	$\mathbb{R}$	Raiz quadrada		$\sqrt{\_}_\epsilon$				
X^*	X^*	Filtragem			filter p	filter p		
X^*	X^*	Ordenação	$bSort$		$iSort$, $sSort$		$mSort$	$qSort$
X^*	X^{**}	Grupos			$chunksOf\ n$			
$X^* \times X^*$	X^*	Junção				merge, $uconc$		
$X \times X^*$	X^*	Inserção				insert		
$\mathbb{B} \times \mathbb{N}_0$	$(\mathbb{N}_0 \times \mathbb{B})^*$	Puzzles						$hanoi$
BTree (X, Y)	$1 + Y$	Look-up		$lookup\ x$				
T (X, Y)	$1 + Y$	Look-up			$lookup\ x$			
T X	T X	Inversão			$reverse$		mirror	mirror
T X	$\mathbb{N}_0$	Cardinalidades			length		$count$	$count$
T X	$\mathbb{N}_0$	Profundidades					$depth$	$depth$
T X	X^*	Travessias					$tips$	$inordt$, $preordt$, $posordt$
T X	X^{**}	Caminhos			$prefixes$, $sufixes$			$traces$
T (T X)	T X	'Multiplicação'			μ		μ	

Cálculo de Programas
Aula T10(b)

PROGRAM DESIGN BY CALCULATION

4

WHY MONADS MATTER

In this chapter we present a powerful device in state-of-the-art functional programming, that of a *monad*. The monad concept is nowadays of primary importance in computing science because it makes it possible to describe computational effects as disparate as input/output, comprehension notation, state variable updating, probabilistic behaviour, context dependence, partial behaviour *etc.* in an elegant and uniform way.

Our motivation to this concept will start from a well-known problem in functional programming (and computing as a whole) — that of coping with undefined computations.

4

WHY MONADS MATTER

In this chapter we present a powerful device in tional programming, that of a *monad*. The monad of primary importance in computing science bec sible to describe computational effects as dispar comprehension notation, state variable updating, probabilistic behaviour, context dependence, partial behaviour *etc.* in an elegant and uniform way.

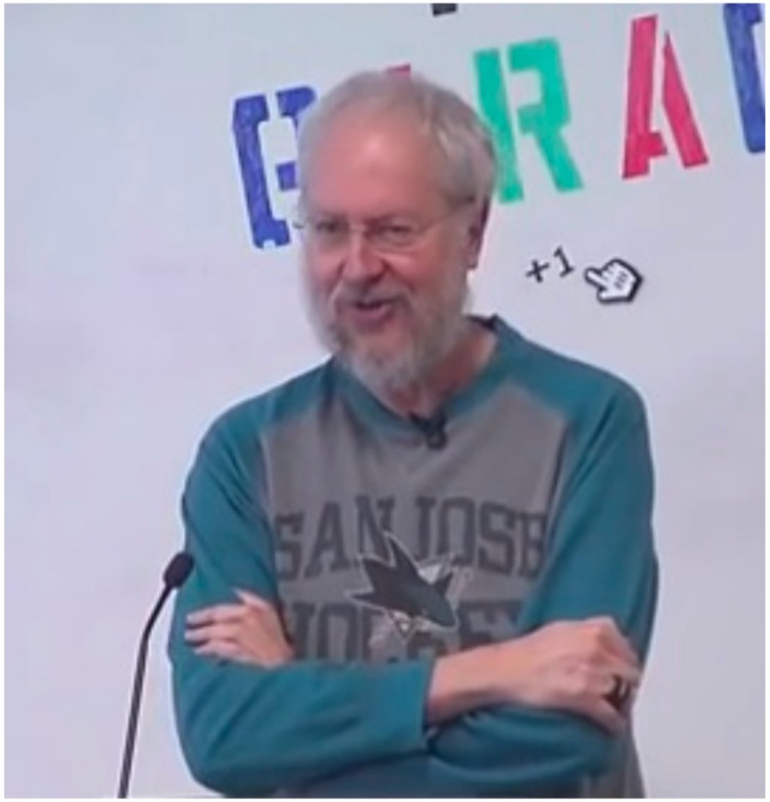
Our motivation to this concept will start from a well-known problem in functional programming (and computing as a whole) — that of coping with undefined computations.


CHANCE

Probability of the sum

"Monads [...] come with a curse. The monadic curse is that once someone learns what monads are and how to use them, they lose the ability to explain it to other people"

(Douglas Crockford: *Google Tech Talk on how to express monads in JavaScript* 2013)



Douglas Crockford (2013)

Partial functions

$$1 + B \xleftarrow{g} A$$

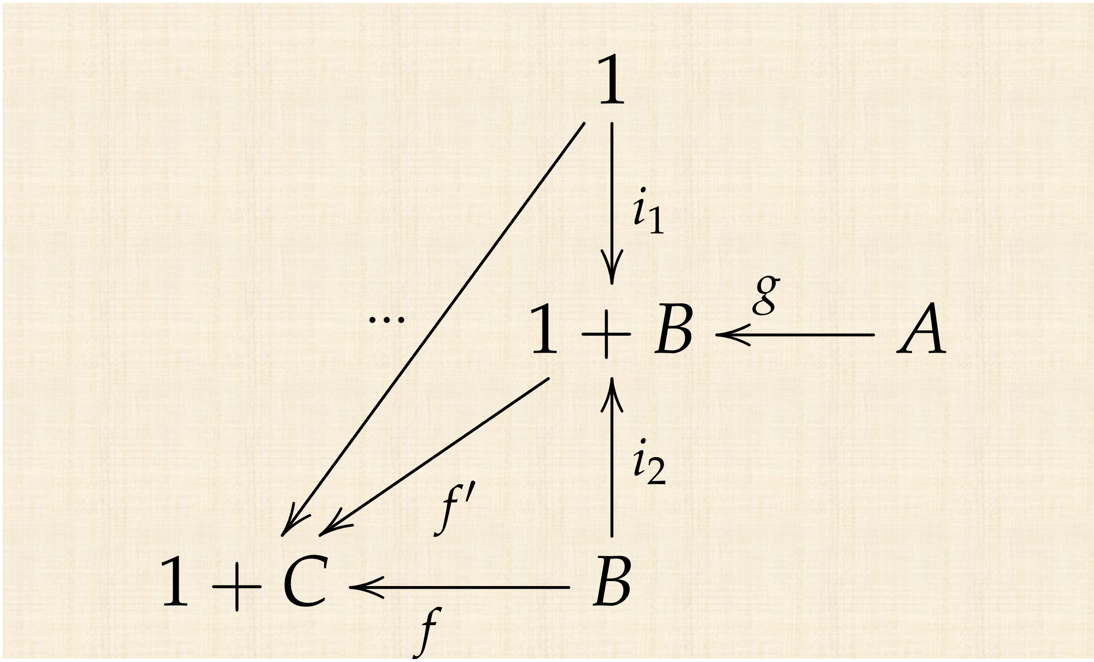
$$B \xleftarrow{f} A$$

$$1 + B \xleftarrow{i_2 \cdot f} A$$

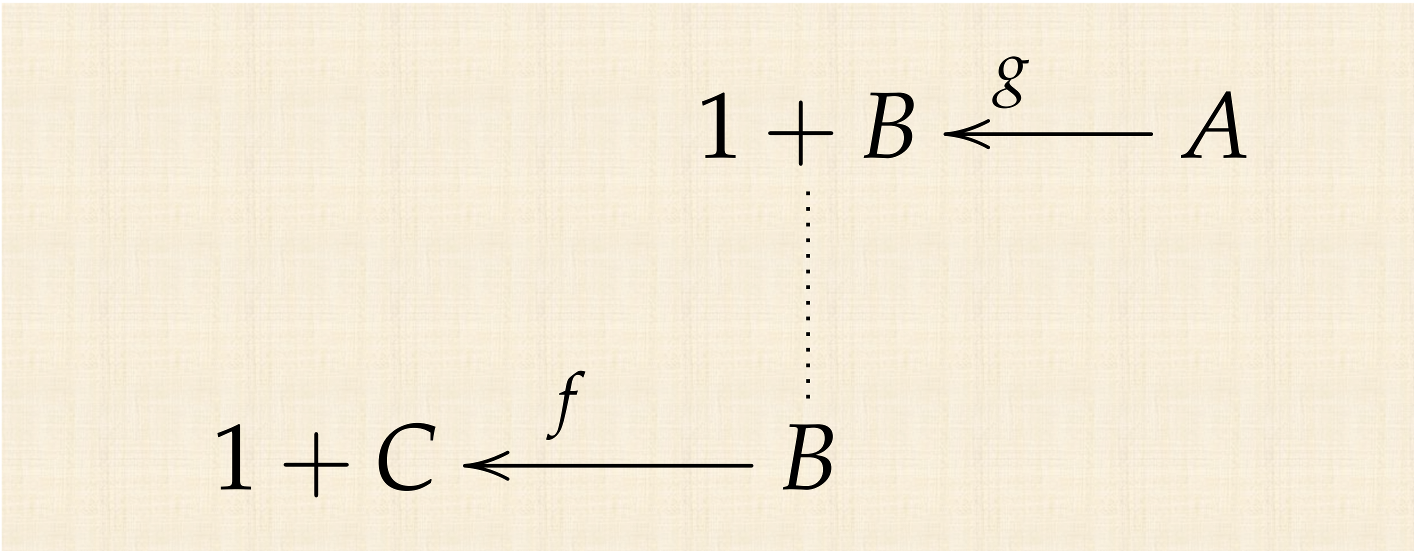
Partial functions

$$\begin{array}{ccccc} & & 1+B & \xleftarrow{\quad g \quad} & A \\ & & \vdots & & \\ 1+C & \xleftarrow{\quad f \quad} & B & & \end{array}$$

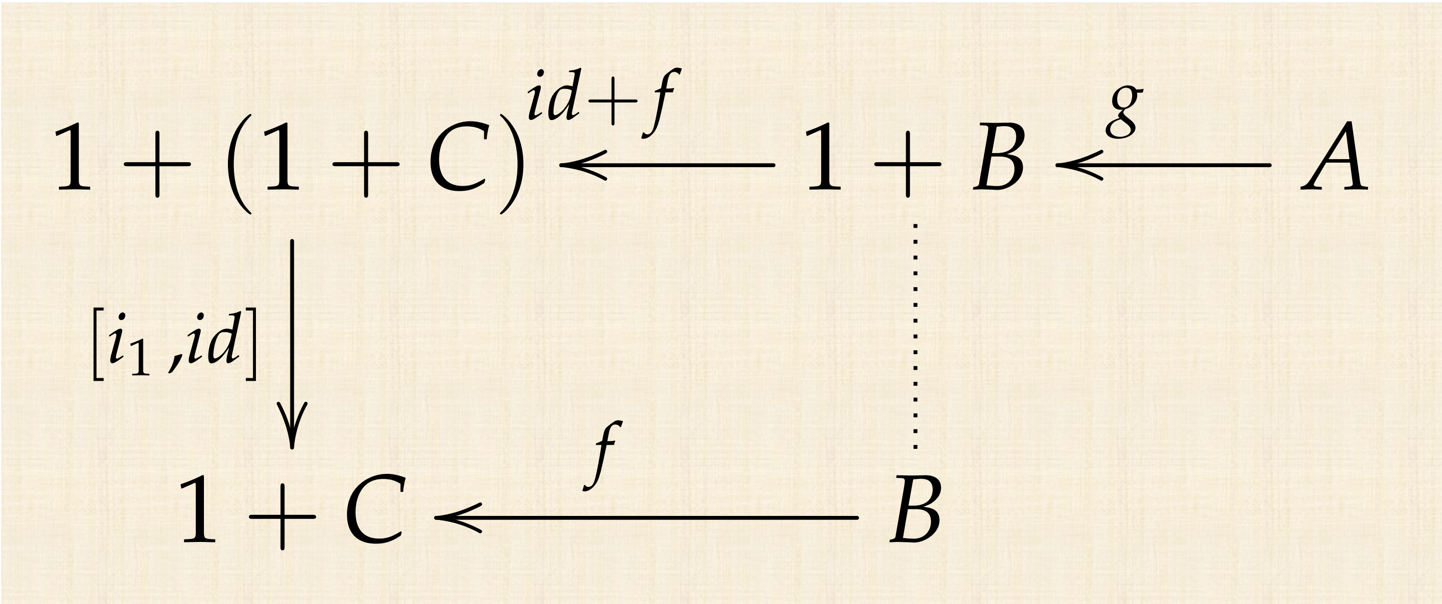
Partial functions



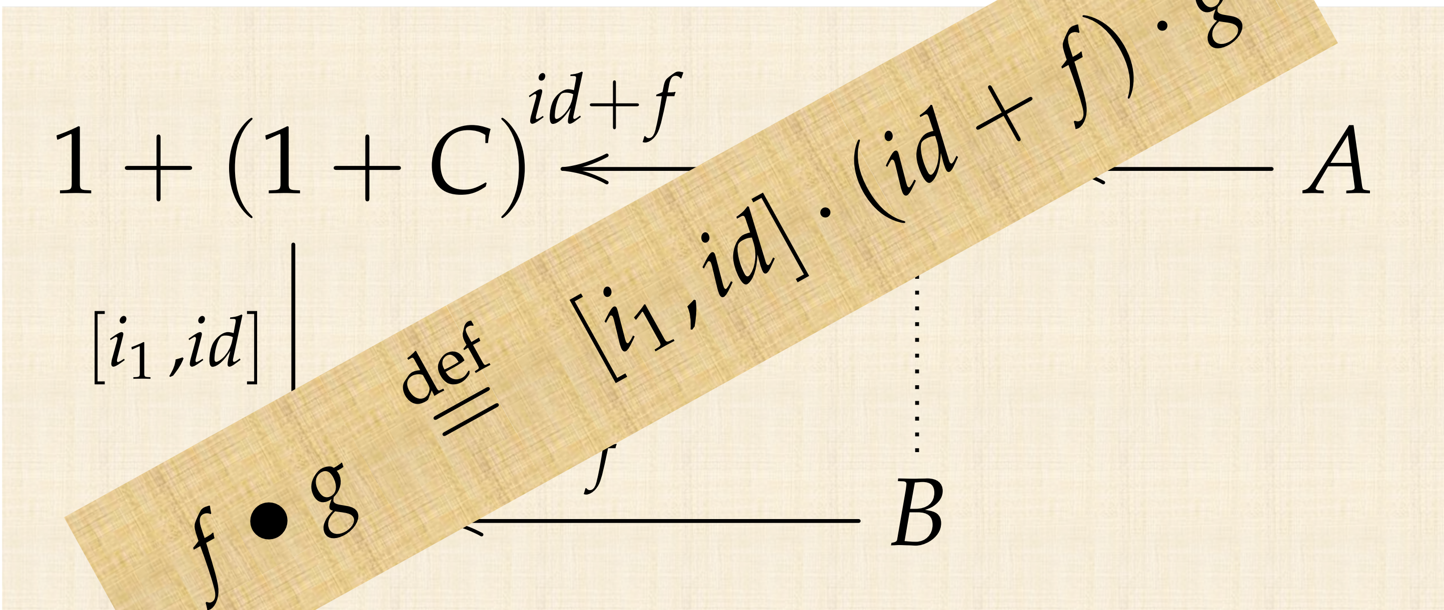
Partial composition



Partial composition

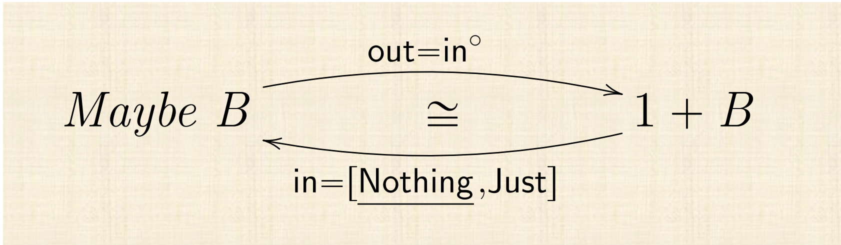


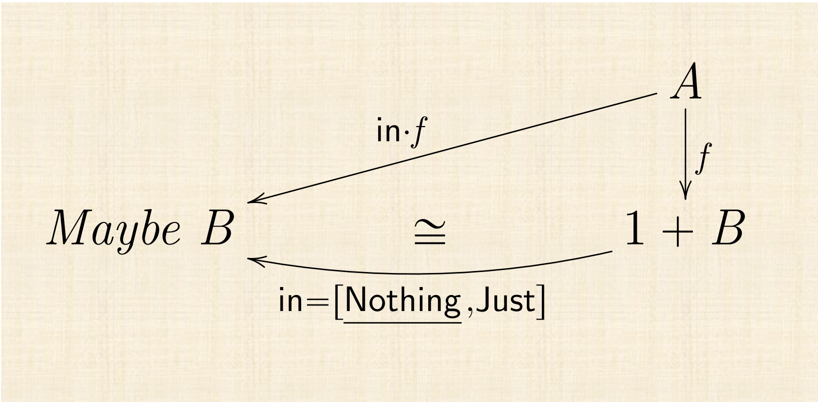
Partial composition



$$f \bullet g \stackrel{\text{def}}{=} [i_1, id] \cdot (id + f) \cdot g$$

‘Maybe functions’

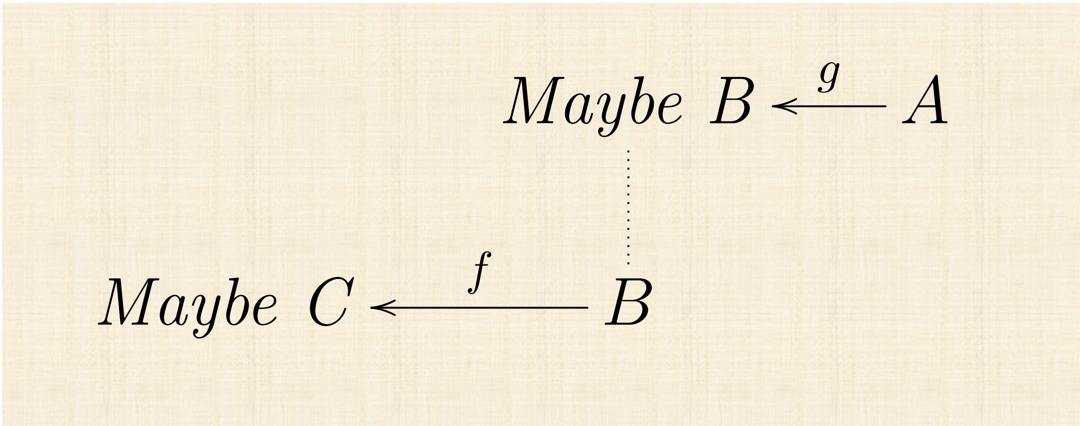




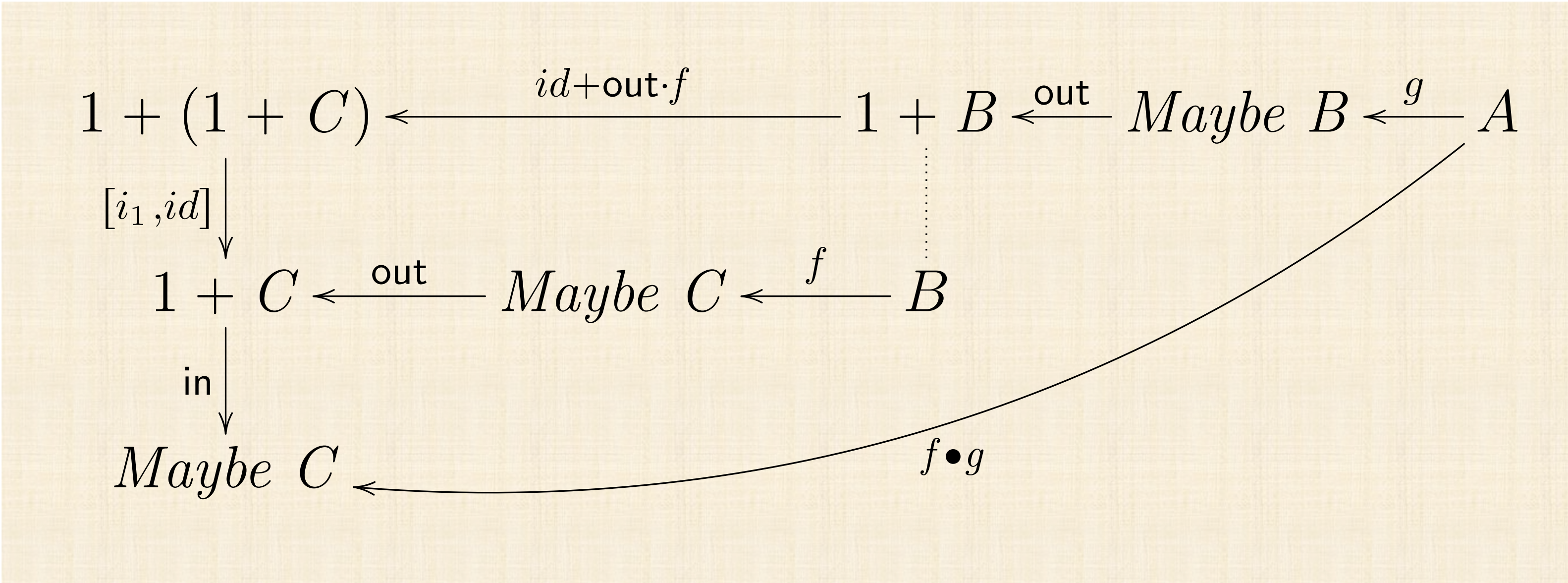
$$\text{out Nothing} = i_1\ ()$$
$$\text{out}\ (\text{Just}\ a) = i_2\ a$$

```haskell
data Maybe a = Nothing | Just a
```

Composing ‘Maybe functions’



Composing ‘Maybe functions’



Composing 'Maybe functions'

$$f \bullet g = \mathsf{in} \cdot [i_1, \mathsf{out} \cdot f] \cdot \mathsf{out} \cdot g$$

$\equiv$ { fusão-+ e $\mathsf{in} \cdot \mathsf{out} = id$ }

$$f \bullet g = [\mathsf{in} \cdot i_1, f] \cdot \mathsf{out} \cdot g$$

$\equiv$ { introdução da variável a }

$$(f \bullet g)\ a = [\mathsf{in} \cdot i_1, f]\ (\mathsf{out}\ (g\ a))$$

$\equiv$ { definição de out }

$$(f \bullet g)\ a = \mathbf{if}\ g\ a = \mathsf{Nothing}\ \mathbf{then}\ [\mathsf{in} \cdot i_1, f]\ (i_1\ ())\ \mathbf{else}\ [\mathsf{in} \cdot i_1, f]\ (i_2\ (g\ a))$$

$\equiv$ { cancelamento-+ e simplificação }

$$(f \bullet g)\ a = \mathbf{if}\ g\ a = \mathsf{Nothing}\ \mathbf{then}\ \mathsf{Nothing}\ \mathbf{else}\ f\ (g\ a)$$

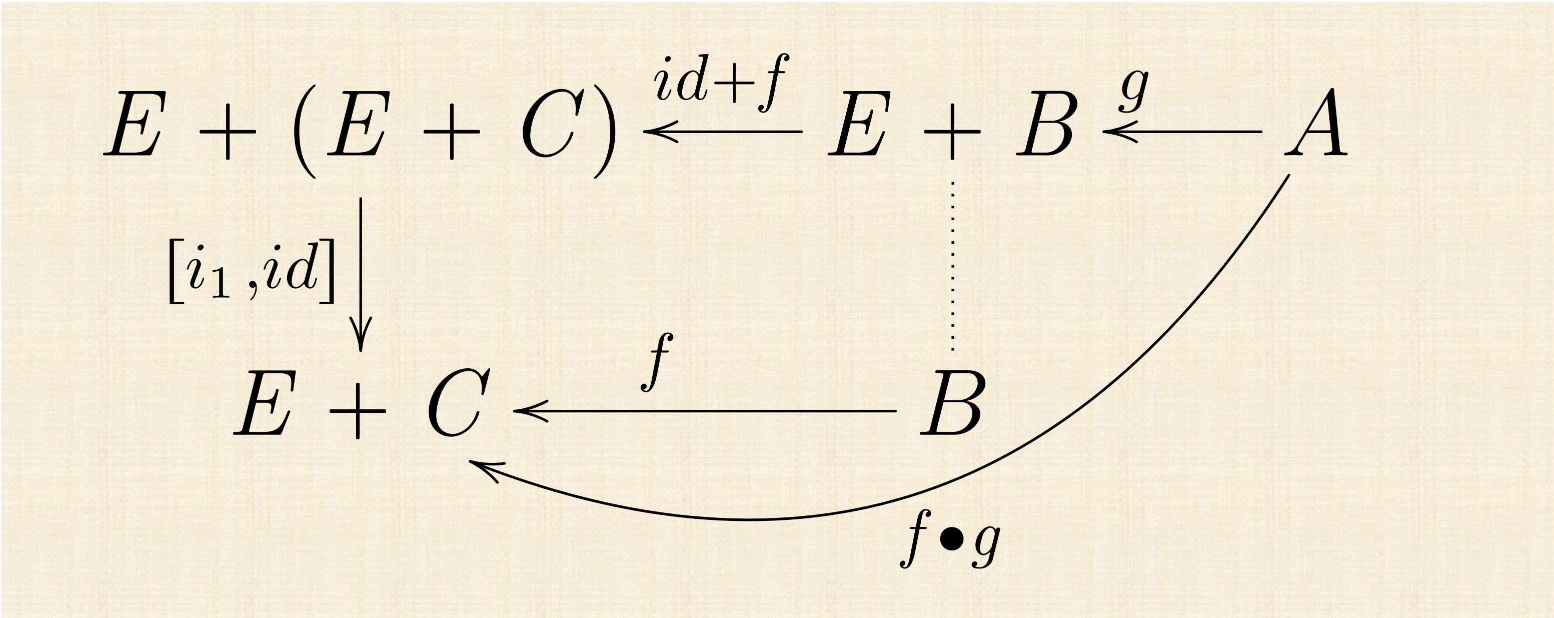
Error messages

$$E + B \xleftarrow{g} A$$

$$B \xleftarrow{f} A$$

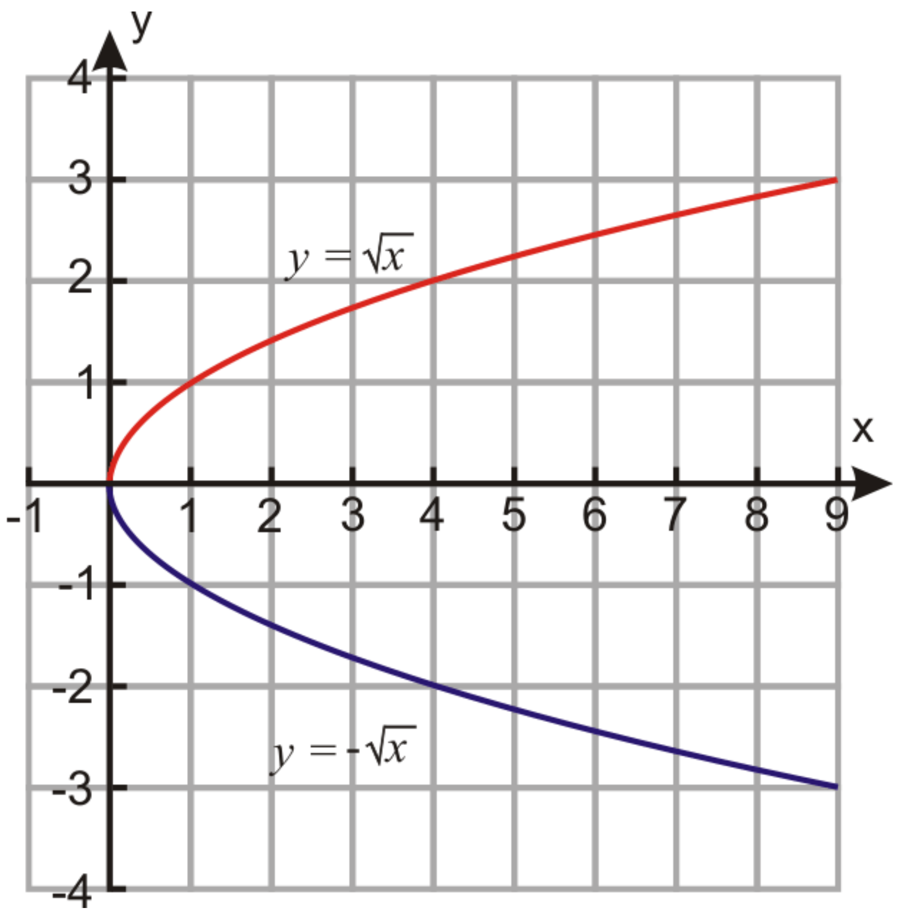
$$E + B \xleftarrow{i_2 \cdot f} A$$

Handling error messages




Unchecked Exception
Checked Exception
Error
Exception Handling

Square root "function"



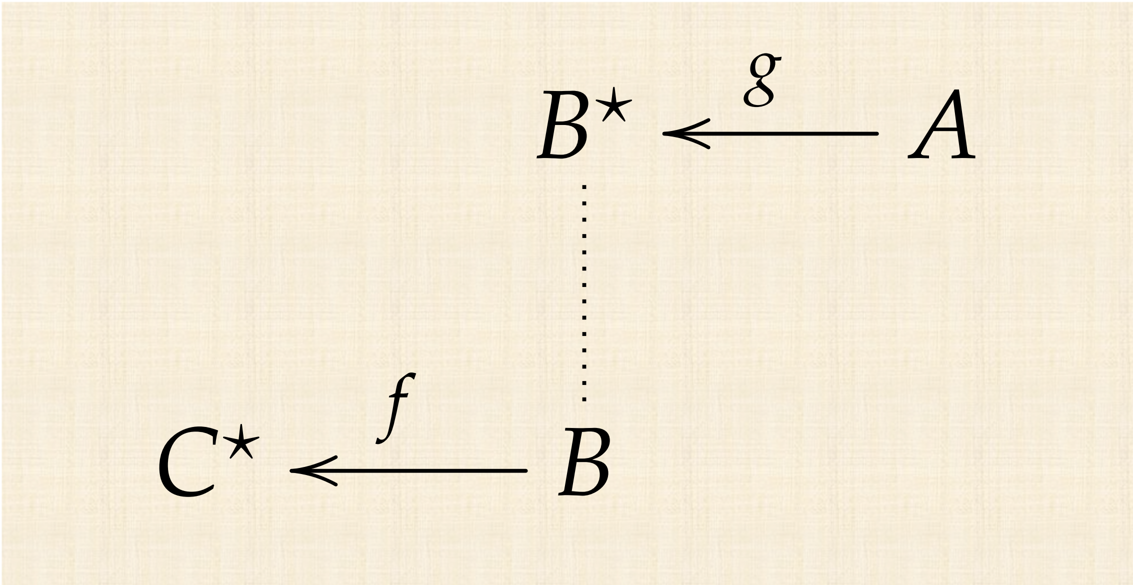
'Undecided' ("nondeterministic") functions

$$B^\star \xleftarrow{\;g\;} A$$

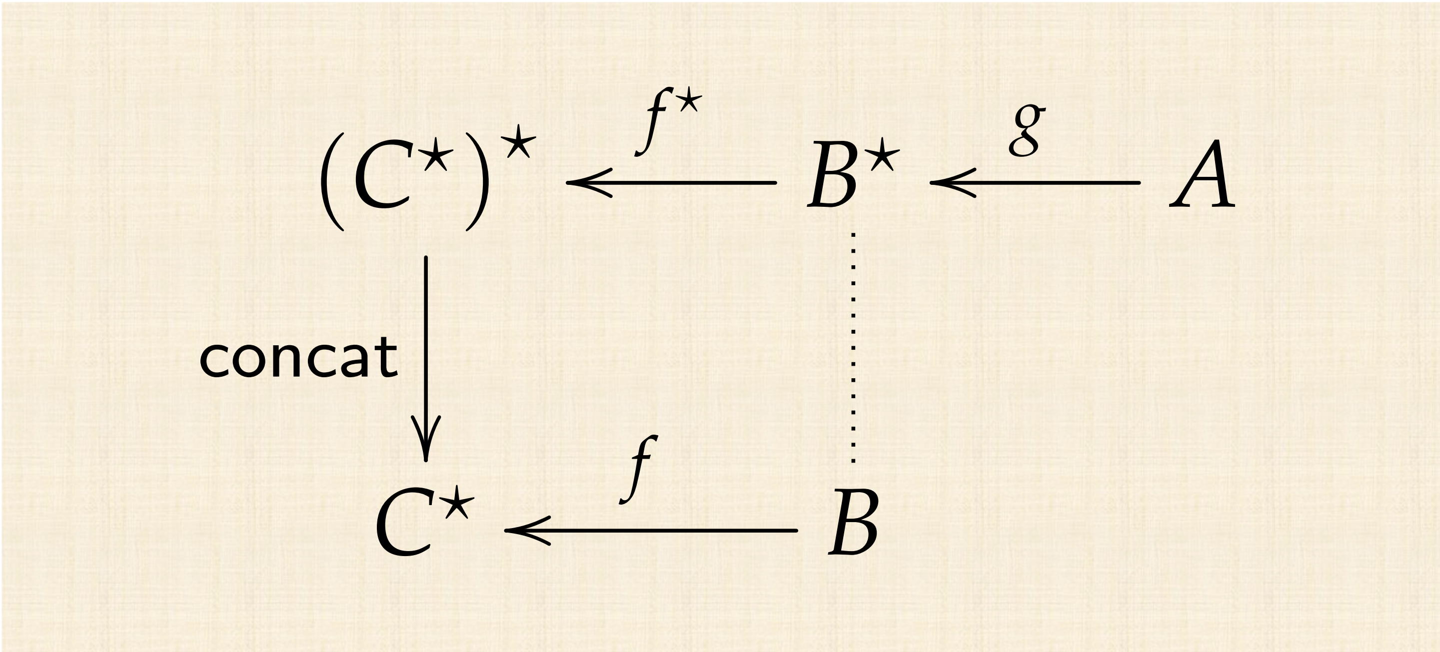
$$B \xleftarrow{\;f\;} A$$

$$B^* \xleftarrow{\;\mathsf{singl} \cdot f\;} A$$

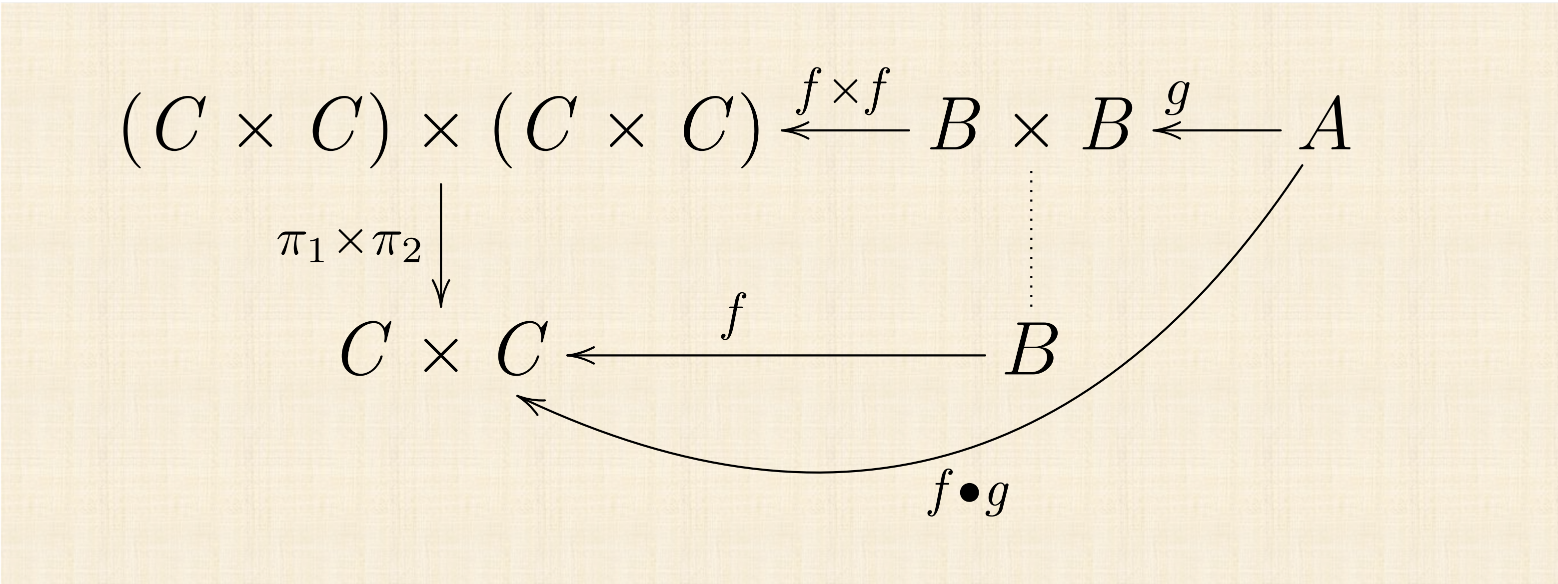
Composing 'undecided' functions



Composing 'undecided' functions



Composing functions that yield pairs



FUNCTORS so far

$$\mathbf{T}\, X = 1 + X$$

$$\mathbf{T}\, X = Maybe\ X$$

$$\mathbf{T}\, X = E + X$$

$$\mathbf{T}\, X = X^*$$

$$\mathbf{T}\, X = X \times X$$

Similar structure

$$X \xrightarrow{i_2} 1 + X \xleftarrow{[i_1, id]} 1 + (1 + X)$$

$$X \xrightarrow{i_2} E + X \xleftarrow{[i_1, id]} E + (E + X)$$

$$X \xrightarrow{\mathsf{Just}} Maybe\ X \xleftarrow{\mu} Maybe\ (Maybe\ X)$$

$$X \xrightarrow{singl} X^* \xleftarrow{\mathsf{concat}} (X^*)^*$$

$$X \xrightarrow{\langle id, id \rangle} X \times X \xleftarrow{\pi_1 \times \pi_2} (X \times X) \times (X \times X)$$

MONAD

$$X \xrightarrow{\;u\;} \mathbf{T}\,X \xleftarrow{\;\mu\;} \mathbf{T}\,(\mathbf{T}\,X)$$

MONAD

$$X \xrightarrow{u} \mathbf{T}\, X \xleftarrow{\mu} \mathbf{T}\,(\mathbf{T}\, X)$$

Unit Multiplication

MONAD

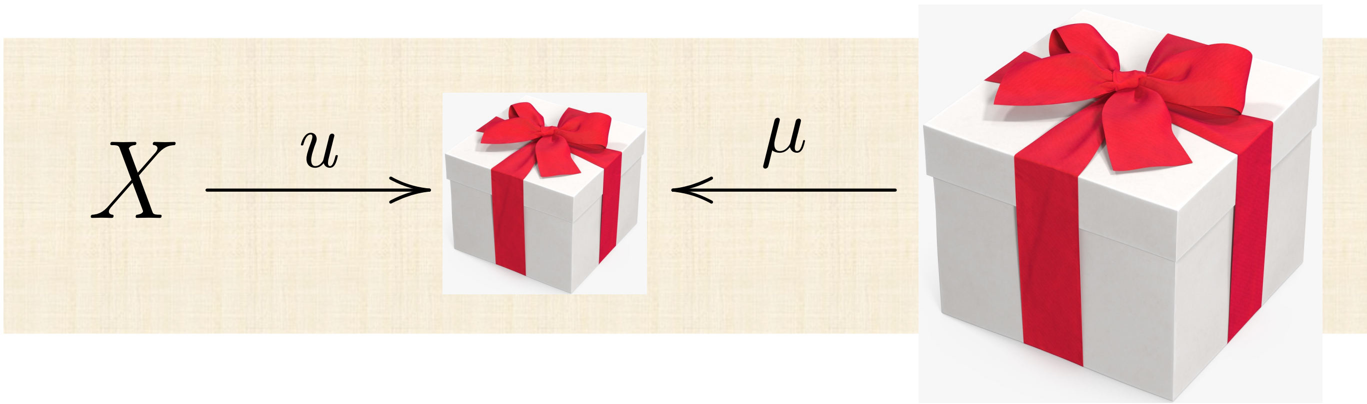
$$X \xrightarrow{u} \mathbf{T}\, X \xleftarrow{\mu} \mathbf{T}\,(\mathbf{T}\, X)$$

Monad = **Functor** + unit + multiplication

Monad = “racing” functor

$$X \xrightarrow{u} \mathbf{T}\,X \xleftarrow{\mu} \mathbf{T}\,(\mathbf{T}\,X)$$

MONAD



Monad = **Functor** + unit + multiplication