Relations among Matrices over a Semiring

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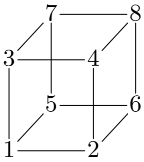
Content

Motivating Example I

A matrix model of a computer network:



$$G = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

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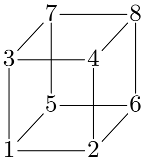

$$G = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

1. G is a Boolean matrix, i.e., a relation.
2. G can be used to calculate qualitative properties/algorithms of the network, e.g., a simple message passing algorithm:

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$B \sqsubseteq G$ is a map so the $p; B^* = \top\!\!\top$ for every point p.

A different matrix model:

$$M = \begin{bmatrix} 0 & \frac{9}{10} & \frac{9}{10} & 0 & \frac{9}{10} & 0 & 0 & 0 \\ \frac{9}{10} & 0 & 0 & \frac{9}{10} & 0 & \frac{9}{10} & 0 & 0 \\ \frac{9}{10} & 0 & 0 & \frac{9}{10} & 0 & 0 & \frac{9}{10} & 0 \\ 0 & \frac{9}{10} & \frac{9}{10} & 0 & 0 & 0 & 0 & \frac{9}{10} \\ \frac{9}{10} & 0 & 0 & 0 & 0 & \frac{9}{10} & \frac{9}{10} & 0 \\ 0 & \frac{9}{10} & 0 & 0 & \frac{9}{10} & 0 & 0 & \frac{9}{10} \\ 0 & 0 & \frac{9}{10} & 0 & \frac{9}{10} & 0 & 0 & \frac{9}{10} \\ 0 & 0 & 0 & \frac{9}{10} & 0 & \frac{9}{10} & \frac{9}{10} & 0 \end{bmatrix}$$

3. M is a matrix over the Bayesian, possibilistic or Viterbi semiring $S = \langle [0 \ldots 1], \max, *, 0, 1 \rangle$.
4. M can be used to calculate quantitative properties of the network, e.g., what is the probability that a message can be send successfully from one node to the other:

Motivating Example III

$$M^* = \begin{bmatrix}
\frac{81}{100} & \frac{9}{10} & \frac{9}{10} & \frac{81}{100} & \frac{9}{10} & \frac{81}{100} & \frac{81}{100} & \frac{729}{1000} \\
\frac{9}{10} & \frac{81}{100} & \frac{81}{100} & \frac{9}{10} & \frac{81}{100} & \frac{9}{10} & \frac{729}{1000} & \frac{81}{100} \\
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\frac{81}{100} & \frac{729}{1000} & \frac{9}{10} & \frac{81}{100} & \frac{9}{10} & \frac{81}{100} & \frac{81}{100} & \frac{9}{10} \\
\frac{729}{1000} & \frac{81}{100} & \frac{81}{100} & \frac{9}{10} & \frac{81}{100} & \frac{9}{10} & \frac{9}{10} & \frac{81}{100}
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\end{bmatrix}$$

5. G is a matrix over the multiplicative idempotent elements $I(S)$ of S.
6. The Hadamard product of matrices over $I(S)$ is the relational meet.

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Goal of this paper/presentation is to find suitable structures so that $I(S)$ becomes a distributive lattice, and, hence, the matrices over $I(S)$ form a distributive allegory.

Definition

A structure $\langle S, +, *, 0, 1\rangle$ is called a semiring iff

1. $\langle S, +, 0\rangle$ is a commutative monoid, i.e., we have
   1. $x + (y + z) = (x + y) + z$ for all $x, y, z \in S$, (Associativity)
   2. $x + 0 = 0 + x = x$ for all $x \in S$, (Identity Law)
   3. $x + y = y + x$ for all $x, y \in S$. (Commutativity)
2. $\langle S, *, 1\rangle$ is a monoid, i.e., we have
   1. $x * (y * z) = (x * y) * z$ for all $x, y, z \in S$, (Associativity)
   2. $x * 1 = 1 * x = x$ for all $x \in S$. (Identity Law)
3. Multiplication left- and right-distributes over addition, i.e., we have
   1. $x * (y + z) = (x * y) + (x * z)$ for all $x, y, z \in S$, (Left Distributivity)
   2. $(x + y) * z = (x * z) + (y * z)$ for all $x, y, z \in S$. (Right Distributivity)
4. Zero is an annihilator for multiplication, i.e., we have
   1. $x * 0 = 0 * x = 0$ for all $x \in S$. (Annihilator Law)

A semiring is called commutative if $*$ is commutative, i.e., if we have $x * y = y * x$ for all $x, y \in S$.

Idempotent Elements

An element of a semiring $x \in S$ is called (multiplicative) idempotent iff $x * x = x^2 = x$. We will denote the set of all (multiplicative) idempotent elements of S by $I(S)$.

Lemma

Let $\langle S, +, *, 0, 1\rangle$ be a commutative semiring. Then $\langle I(S), *, 0, 1\rangle$ is a semilattice with least element 0 and greatest element 1.

If $M = [a_{ij}]_{mn}$ denotes a matrix of size $m \times n$ with coefficients a_{ij} from S, then we define

$$[a_{ij}]_{mn} + [b_{ij}]_{mn} = [a_{ij} + b_{ij}]_{mn}$$

and we have

$$(M+N)+P = M+(N+P), \quad M+N = N+M, \quad M+\perp\!\!\!\perp = \perp\!\!\!\perp+M = M,$$

where $\perp\!\!\!\perp = [0]_{mn}$ is the matrix with 0's everywhere.

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Furthermore, if two finite matrices are of appropriate size, i.e., $M = [a_{ij}]_{mn}$ and $N = [b_{jk}]_{np}$, then matrix multiplication can be defined as usual by

$$[a_{ij}]_{mn}[b_{jk}]_{np} = [\sum_{j=1}^{n} a_{ij} * b_{jk}]_{mp}.$$

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Matrix multiplication together with the identity matrix forms a category.

Furthermore, we have $M\perp\!\!\!\perp = \perp\!\!\!\perp = \perp\!\!\!\perp M$ and matrix multiplication is bilinear, i.e., we have

$$M(N+P) = MN + MP, \qquad (N+P)Q = NQ + PQ.$$

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Last but not least, we may also define the converse (or transpose) of a matrix and the Hadamard product of matrices of equal size by

$$[a_{ij}]^{\smile}_{mn} = [a_{ji}]_{nm}, \qquad [a_{ij}]_{mn} \cdot [b_{ij}]_{mn} = [a_{ij} * b_{ij}]_{mn}.$$

Converse distributes over $+$ and we have $(M\ N)^{\smile} = N^{\smile}\ M^{\smile}$.

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If S is a commutative semiring, then so are the matrices of size $m \times n$ with respect to the matrix sum, the Hadamard product, $\perp\!\!\!\perp$ and $\top\!\!\!\top = [1]_{mn}$.

Matrices over Semirings II

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If S is a commutative semiring, then so are the matrices of size $m \times n$ with respect to the matrix sum, the Hadamard product, $\perp\!\!\!\perp$ and $\top\!\!\!\top = [1]_{mn}$.

A matrix is idempotent iff all coefficients are.

Flattening I

Definition

Let $\langle S, +, *, 0, 1\rangle$ be a commutative semiring. An operation $(.)'$ is called a flattening operation iff

1. $x' * x' = x'$ for all $x \in S$,
2. $x * z = x$ iff $x' * z = x'$ for all $z \in I(S)$.

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If $z \in$(S), then $x' * z = x' \iff x' \leq z$, i.e., the operation $(.)'$ assigns to an $x \in S$ the smallest idempotent element z so that $x * z = x$.

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Lemma

Let $(.)'$ be a flattening operation. Then we have:

1. $x' = x$ *iff* $x \in I(S)$.
2. $x'' = x'$.

Flattening II

Lemma

1. *A flattening operation is unique.*
2. *If $I(S) = \{0, 1\}$, then the canonical flattening operation*

$$x' := \begin{cases} 1 & \textit{iff } x \neq 0, \\ 0 & \textit{iff } x = 0, \end{cases}$$

is a flattening operation.

Lemma

Suppose $\langle S, +, *, 0, 1\rangle$ is a commutative semiring.

1. *If S is a ring, i.e., a semiring with additive inverses, then $I(S)$ is a distributive lattice with $x \vee y = x + y - x * y$.*
2. *If $+$ satisfies the absorption law $x + x * y = x$ for all $x, y \in S$, then $I(S)$ is a distributive lattice with $x \vee y = x + y$.*
3. *If S is multiplicative cancelative, i.e., $x * y = x * z$ implies $y = z$ for every $x \neq 0$, then $I(S) = \{0, 1\}$ is the Boolean algebra with two elements.*

Definition

A structure $\langle D, +, *, \sqcup, 0, 1\rangle$ is called a sup-semiring iff

1. $\langle D, +, *, 0, 1\rangle$ is a commutative semiring.
2. $\langle D, \sqcup\rangle$ is a commutative semigroup, i.e., we have
    1. $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$ for all $x, y, z \in D$, (Associativity)
    2. $x \sqcup y = y \sqcup x$ for all $x, y \in D$, (Commutativity)
3. $(x \sqcup y) * (x \sqcup y) = x \sqcup y$ for all $x, y \in D$, (Relative Idempotency)
4. $x * (x \sqcup y) = x$ for all $x, y \in D$, (Absorption)
5. if $x * x = x$, then $x \sqcup (x * y) = x$ for all $x, y \in D$, (Relative Absorption)
6. if $x * x = x$ and $y * y = y$ and $z * z = z$, then $x * (y \sqcup z) = x * y \sqcup x * z$ for all $x, y, z \in D$. (Relative Distributivity)

Lemma

1. *The following two conditions are equivalent:*
   1. $x * y = x$ *and* y *idempotent,*
   2. $x \sqcup y = y$.
2. x *is idempotent iff* $x \sqcup x = x$.
3. $x \sqcup z = z$ *and* $y \sqcup z = z$ *implies* $z * (x \sqcup y) = x \sqcup y$.
4. $x \sqcup 0 = x \sqcup x$.
5. $0 \sqcup 0 = 0$ *and* $x \sqcup 1 = 1$.
6. $x \sqcup y = (x \sqcup 0) \sqcup (y \sqcup 0)$.

Theorem

Let $\langle D, +, *, \sqcup, 0, 1\rangle$ be a sup-semiring. Then the idempotent elements, i.e., the structure $\langle I(D), *, \sqcup, 0, 1\rangle$, form a distributive lattice. Furthermore, $x' = x \sqcup 0$ is a flattening operation for $\langle D, +, *, 0, 1\rangle$.

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Theorem

Let $\langle S, +, *, 0, 1\rangle$ be a commutative semiring with a flattening operation. Furthermore, assume that $\langle I(S), \vee, *, 0, 1\rangle$ is a distributive lattice. Then $\langle S, +, *, \sqcup, 0, 1\rangle$ with $x \sqcup y = x' \vee y'$ is a sup-semiring.

Theorem

Let $\langle D, +, *, \sqcup, 0, 1\rangle$ be a sup-semiring. Then the idempotent elements, i.e., the structure $\langle I(D), *, \sqcup, 0, 1\rangle$, form a distributive lattice. Furthermore, $x' = x \sqcup 0$ is a flattening operation for $\langle D, +, *, 0, 1\rangle$.

Theorem

Let $\langle S, +, *, 0, 1\rangle$ be a commutative semiring with a flattening operation. Furthermore, assume that $\langle I(S), \vee, *, 0, 1\rangle$ is a distributive lattice. Then $\langle S, +, *, \sqcup, 0, 1\rangle$ with $x \sqcup y = x' \vee y'$ is a sup-semiring.

Lemma

In the context of the other axioms in Def. 5 rel. distributivity is equivalent to

$$x' * (y \sqcup z) = x' * y' \sqcup x' * z'.$$

Lemma

The theory of sup-semirings is not equational, i.e., the class of sup-semirings does not form a variety.

Consider the semiring $\langle \mathbb{N}, +, *, 0, 1\rangle$ with the operation

$$x \sqcup y := \begin{cases} 0, & \text{iff } x = y = 0, \\ 1, & \text{otherwise.} \end{cases}$$

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Let $\equiv$ be the equivalence relation that has the three equivalence classes $[0] = \{0\}, [1] = \{1\}$ and $[n] = \{n \in \mathbb{N} \mid n > 1\}$. It is easy to see that $\equiv$ is a congruence and that the induced operations on the equivalence classes are

$+$	$[0]$	$[1]$	$[n]$
$[0]$	$[0]$	$[1]$	$[n]$
$[1]$	$[1]$	$[n]$	$[n]$
$[n]$	$[n]$	$[n]$	$[n]$

$*$	$[0]$	$[1]$	$[n]$
$[0]$	$[0]$	$[0]$	$[0]$
$[1]$	$[0]$	$[1]$	$[n]$
$[n]$	$[0]$	$[n]$	$[n]$

$\sqcup$	$[0]$	$[1]$	$[n]$
$[0]$	$[0]$	$[1]$	$[1]$
$[1]$	$[1]$	$[1]$	$[1]$
$[n]$	$[1]$	$[1]$	$[1]$

Now, $[n]$ is idempotent but $[n] \sqcup [n] * [0] = [n] \sqcup [0] = [1] \neq [n]$, i.e., rel. absorption is not true.

Matrices over Sup-Semirings

We define

$$[a_{ij}]_{mn} \sqcup [b_{ij}]_{mn} = [a_{ij} \sqcup b_{ij}]_{mn}, \qquad [a_{ij}]_{mn};[b_{jk}]_{np} = [\bigsqcup_{j=1}^{n} a_{ij} * b_{jk}]_{mp}.$$

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With these operations the collection of relations forms a distributive allegory.

We define

$$[a_{ij}]_{mn} \sqcup [b_{ij}]_{mn} = [a_{ij} \sqcup b_{ij}]_{mn}, \qquad [a_{ij}]_{mn}; [b_{jk}]_{np} = [\bigsqcup_{j=1}^{n} a_{ij} * b_{jk}]_{mp}.$$

With these operations the collection of relations forms a distributive allegory.

Lemma

Suppose Q is a matrix over a sup-semiring. Then we have $Q; \mathbb{I} = \mathbb{I}; Q = Q'$ where $Q' = Q \sqcup [0]$.

Biproducts

Theorem

Suppose $A+B$ *together with* $\iota : A \to A+B$ *and* $\kappa : B \to A+B$ *is a relational sum of* A *and* B*. Then* $A+B$ *together with* $\iota^{\smile}, \kappa^{\smile}, \iota, \kappa$ *is a biproduct with respect to* $+$ *and linear composition.*

Future Work

1. Provide an categorical/algebraic structure for matrices over a sup-semiring.
2. Study the basic theory of these algebraic structures.
3. Investigate a pseudo-representation theorem similar to the matrix representation for allegories.
4. Apply the theory to real world examples.
5. ...

Thank you
for your attention!

Questions?