Investigating and Computing Bipartitions with Algebraic Means

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Introduction

Contents:

- We use Dedekind categories as an algebraic structure for set-theoretic relations without complements.
- We present purely algebraic definitions of "to be bipartite" and "to possess no odd cycles".
- We prove that both notions coincide.

This generalises D. Kőnig's well-known theorem (Mathematische Annalen 77, pp. 453-465, 1916)

- from undirected graphs to abstract relations,
- to models such as L-relations that are different from set-theoretic relations.

For set-theoretic relations the proof immediately leads to an algorithm.

Dedekind Categories

Specific categories with typed relations $R : X \leftrightarrow Y$ as morphisms, which generalise relation algebra by using residuals instead of complements.

Axioms:

- Complete distributive lattice for union, intersection, ordering, empty and universal relation.
- Associativity of composition and that identity relations are neutral.
- Monotonicity of transposition.
- $(R^{\mathsf{T}})^{\mathsf{T}} = R$ and $(R;S)^{\mathsf{T}} = S^{\mathsf{T}};R^{\mathsf{T}}$.
- **Modular law** $Q;R \cap S \subseteq Q;(R \cap Q^{\mathsf{T}};S)$.
- $Q;R \subseteq S$ if and only if $Q \subseteq S/R$.

Most of the well-known complement-free relation-algebraic rules already hold in a Dedekind category. R^* denotes **reflexive-transitive closure** and R^+ denotes **transitive closure**.

The Main Result

Definition (Bipartition)

Given a (homogeneous) relation R, then (v, w) is a **bipartition** of R if

- v, w **vectors**, i.e., $v;\mathsf{L} = v$ and $w;\mathsf{L} = w$.
- $v \cap w = \mathsf{O}$,
- $R \subseteq v;w^\mathsf{T} \cup w;v^\mathsf{T}$.

If there exists a bipartition, then R is called **bipartite**.

If R, v and w are from a relation algebra (i..e., complements exists), then each bipartition (v, w) of R leads to the bipartition $(v, \overline{v})$ of R.

Theorem (Kőnig's theorem on Dedekind categories)

Let R be a relation. Then we have:

$$\boxed{R \text{ bipartite} \iff R;(R;R)^* \cap \mathsf{I} = \mathsf{O}}$$

The First Direction

Lemma 1 (Properties of disjoint vectors)

Let v and w be vectors with $v \cap w = \mathsf{O}$. Then we have:

(1) $v^{\mathsf{T}};w = \mathsf{O}$ and $w^{\mathsf{T}};v = \mathsf{O}$.

(2) $v;v^{\mathsf{T}} \cup w;w^{\mathsf{T}}$ is **transitive**.

(3) $(v;w^{\mathsf{T}} \cup w;v^{\mathsf{T}});(v;w^{\mathsf{T}} \cup w;v^{\mathsf{T}}) \subseteq v;v^{\mathsf{T}} \cup w;w^{\mathsf{T}}$.

Theorem 2 (Bipartitions consist of stable sets)

Let R be a relation and v, w be vectors with $v \cap w = \mathsf{O}$. Then we have:

$$R \subseteq v;w^{\mathsf{T}} \cup w;v^{\mathsf{T}} \iff \begin{cases} R;v \subseteq w \ \wedge \\ R;w \subseteq v \ \wedge \\ R \subseteq (v \cup w);(v \cup w)^{\mathsf{T}} \end{cases}$$

Theorem 3 (Bipartite implies no odd cycles)

Let R be a relation and v, w be vectors. Then we have:

$$(v,w) \text{ bipartition of } R \implies R;(R;R)^* \cap \mathsf{I} = \mathsf{O}$$

Proof: Using the modular law in the first step, we obtain

$$v;w^{\mathsf{T}} \cap \mathsf{I} \subseteq v;(w^{\mathsf{T}} \cap v^{\mathsf{T}};\mathsf{I}) = v;(v \cap w)^{\mathsf{T}} = \mathsf{O},$$

and $w;v^{\mathsf{T}} \cap \mathsf{I} = \mathsf{O}$ follows similarly. Now, we get the claim as follows:

$$\begin{array}{rll} R;(R;R)^* \cap \mathsf{I} & \subseteq R;((v;w^{\mathsf{T}} \cup w;v^{\mathsf{T}});(v;w^{\mathsf{T}} \cup w;v^{\mathsf{T}}))^* \cap \mathsf{I} & \text{assumption} \\ & \subseteq R;(v;v^{\mathsf{T}} \cup w;w^{\mathsf{T}})^* \cap \mathsf{I} & \text{Lem. 1 (3)} \\ & = R;(\mathsf{I} \cup (v;v^{\mathsf{T}} \cup w;w^{\mathsf{T}})^+) \cap \mathsf{I} & \text{property clos.} \\ & = R;(\mathsf{I} \cup v;v^{\mathsf{T}} \cup w;w^{\mathsf{T}}) \cap \mathsf{I} & \text{Lem. 1 (2)} \\ & = (R \cup R;v;v^{\mathsf{T}} \cup R;w;w^{\mathsf{T}}) \cap \mathsf{I} & \\ & \subseteq (R \cup w;v^{\mathsf{T}} \cup v;w^{\mathsf{T}}) \cap \mathsf{I} & \text{Thm. 2 ``}\Rightarrow\text{''} \\ & = (w;v^{\mathsf{T}} \cup v;w^{\mathsf{T}}) \cap \mathsf{I} & \text{assumption} \\ & = (w;v^{\mathsf{T}} \cap \mathsf{I}) \cup (v;w^{\mathsf{T}} \cap \mathsf{I}) & \\ & = \mathsf{O} & \text{aux. results} \end{array}$$

The Remaining Direction: Problem Reduction

Theorem 4 (Reduction to **symmetric** relations)
Let R be a relation and v, w be vectors. Then we have:

$$R \subseteq v;w^\mathsf{T} \cup w;v^\mathsf{T} \iff R \cup R^\mathsf{T} \subseteq v;w^\mathsf{T} \cup w;v^\mathsf{T}$$

Hence, for all relations R it suffices to prove:

$$R = R^\mathsf{T} \wedge R;(R;R)^* \cap \mathsf{I} = \mathsf{O} \implies R \text{ bipartite}$$

But symmetry of R implies symmetry of its reflexive-transitive closure R^*.
Hence, for all relations R it suffices to prove:

$$R^* = (R^*)^\mathsf{T} \wedge R;(R;R)^* \cap \mathsf{I} = \mathsf{O} \implies R \text{ bipartite}$$

Theorem 5 (Main theorem for the remaining direction)

Let R be a relation, $R^* = (R^*)^\mathsf{T}$ and u be a vector such that

$$(a) \quad R;(R;R)^*;u \cap u = \mathsf{O} \qquad\qquad (b) \quad R \subseteq R^*;u;u^\mathsf{T};R^*.$$

Then (v, w) is a bipartition of R if we define v, w as follows:

$$v := (R;R)^*;u \qquad\qquad w := R;v = R;(R;R)^*;u$$

Graph-theoretic interpretations:

- Condition (a): No vertices of the set modeled by u are connected by an odd path.
- Condition (b): If (x, y) is an arc, then both vertices are reachable from the set modeled by u.
- Definition of v: Models the set of vertices which are reachable from the set modeled by u via an even path.
- Definition of w: Models the set of vertices which are reachable from the set modeled by u via an odd path.

Hence, for all relations R it suffices to prove:

$$\left.\begin{array}{r} R^* = (R^*)^{\mathsf{T}} \;\wedge \\ R;(R;R)^* \cap \mathsf{I} = \mathsf{O} \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} \exists\, u \,:\, u = u;\mathsf{L} \;\wedge \\ \qquad : R;(R;R)^*;u \cap u = \mathsf{O} \;\wedge \\ \qquad : R \subseteq R^*;u;u^{\mathsf{T}};R^* \end{array}\right.$$

It is remarkable that also the converse of Theorem 5 is valid such that, in general, we have the following characterisation of bipartite relations:

Theorem 6 (Characterisation)

Let R be a relation and $R^* = (R^*)^{\mathsf{T}}$. Then we have:

$$R \text{ bipartite } \Longleftrightarrow \left\{\begin{array}{l} \exists\, u \,:\, u = u;\mathsf{L} \;\wedge \\ \qquad : R;(R;R)^*;u \cap u = \mathsf{O} \;\wedge \\ \qquad : R \subseteq R^*;u;u^{\mathsf{T}};R^* \end{array}\right.$$

The Remaining Direction: Solution

The idea in terms of graphs;

- Let $R : X \leftrightarrow X$ be the symmetric adjacency relation of an undirected graph $G = (X, E)$ without odd cycles. Then:

$$R;(R;R)^* \cap \mathsf{I} = \mathsf{O}$$

- From $R = R^\mathsf{T}$ we get $R^* = (R^*)^\mathsf{T}$ such that:

$$R^* \text{ is an } \textbf{equivalence relation}$$

- Consider X/R^*, i.e., the set of **connected components** of G.
- Select from each connected component a single vertex and combine all these vertices to a subset U of X.
- If u is a vector that models U as subset of X, then it fulfills (a) and (b) of Theorem 5 and we are done.

Means for the selectipn of the vertices from the connected components:

Axiom 7 (Relational axiom of choice)

For all relations R there exists a relation F such that:

(1) $F^{\mathsf{T}};F \subseteq \mathsf{I}$ (**univalent**) (2) $F \subseteq R$ (3) $F;\mathsf{L} = R;\mathsf{L}$

Theorem 8 (P. Freyd, A. Scedrov, M. Winter)

For each (partial) equivalence relation P there exists a relation S with

(1) $S;S^{\mathsf{T}} = P$ (2) $S^{\mathsf{T}};S = \mathsf{I}$

and all relations with these properties are isomorphic.

S is called a **splitting** of P and if P is a set-theoretic equivalence relation on X, then the **canonical epimorphism** $\pi : X \to X/P$ is a splitting.

Lemma 9

Assume Axiom 7 to be true and let R be a relation such that $R^* = (R^*)^\mathsf{T}$. Then there exist

- a splitting S of R^*,
- a relation F with $F^\mathsf{T};F \subseteq \mathsf{I}$, $\mathsf{I} \subseteq F;F^\mathsf{T}$ (**total**) and $F \subseteq S^\mathsf{T}$.

Theorem 10 (Existence of u)

Assume Axiom 7 to be true and let R be a relation such that

$$R^* = (R^*)^\mathsf{T} \qquad\qquad R;(R;R)^* \cap \mathsf{I} = \mathsf{O}.$$

If S is a splitting of R^* and F a **mapping** such that $F \subseteq S^\mathsf{T}$, then we get

$$(a)\quad R;(R;R)^*;u \cap u = \mathsf{O} \qquad\qquad (b)\quad R \subseteq R^*;u;u^\mathsf{T};R^*$$

if we define the vector u as $u := F^\mathsf{T};\mathsf{L}$.

Proof: To prove (a) we start with

$$
\begin{aligned}
F;R;(R;R)^*;F^\mathsf{T} &\subseteq F;R^*;F^\mathsf{T} && \text{property clos.}\\
&= F;S;S^\mathsf{T};F^\mathsf{T} && S \text{ splitting of } R^*\\
&\subseteq S^\mathsf{T};S;S^\mathsf{T};S && F \subseteq S^\mathsf{T}\\
&= \mathsf{I};\mathsf{I} && S \text{ splitting}\\
&\subseteq F;F^\mathsf{T} && F \text{ total}
\end{aligned}
$$

Now, (a) can be shown as follows:

$$
\begin{aligned}
R;(R;R)^*;u \cap u &= R;(R;R)^*;F^\mathsf{T};\mathsf{L} \cap F^\mathsf{T};\mathsf{L} && \text{definition } u\\
&= F^\mathsf{T};\mathsf{L} \cap R;(R;R)^*;F^\mathsf{T};\mathsf{L} &&\\
&\subseteq F^\mathsf{T};(\mathsf{L} \cap F;R;(R;R)^*;F^\mathsf{T};\mathsf{L}) && \text{modular law}\\
&= F^\mathsf{T};F;R;(R;R)^*;F^\mathsf{T};\mathsf{L} &&\\
&= F^\mathsf{T};(F;R;(R;R)^*;F^\mathsf{T} \cap F;F^\mathsf{T});\mathsf{L} && \text{aux. result}\\
&= F^\mathsf{T};F;(R;(R;R)^* \cap \mathsf{I});F^\mathsf{T};\mathsf{L} && F \text{ univalent}\\
&= \mathsf{O} && \text{assumption}
\end{aligned}
$$

Verification of (b):

$$
\begin{array}{rll}
R & \subseteq R^* & \text{property clos.} \\
& = S;\mathsf{I};S^\mathsf{T} & S \text{ splitting of } R^* \\
& \subseteq S;F;F^\mathsf{T};F;F^\mathsf{T};S^\mathsf{T} & F \text{ total} \\
& \subseteq S;S^\mathsf{T};F^\mathsf{T};F;S;S^\mathsf{T} & \text{as } F \subseteq S^\mathsf{T} \\
& = R^*;F^\mathsf{T};F;R^* & S \text{ splitting} \\
& \subseteq R^*;F^\mathsf{T};\mathsf{L};F;R^* & \\
& = R^*;F^\mathsf{T};\mathsf{L};(F^\mathsf{T};\mathsf{L})^\mathsf{T};R^* & \\
& = R^*;u;u^\mathsf{T};R^* & \text{definition } u
\end{array}
$$

Theorem 11 (No odd cycles implies bipartite)

Assume Axiom 7 to be true and let R be a relation. Then we have:

$$\boxed{R;(R;R)^* \cap \mathsf{I} = \mathsf{O} \implies R \text{ is bipartite}}$$

Computing Bipartitions

Assumption:

- A set-theoretic relation $R : X \leftrightarrow X$ on a finite set X with symmetric R^* and $R;(R;R)^* \cap \mathsf{I} = \mathsf{O}$.

Goal:

- A relational program that computes a vector $v : X \leftrightarrow \mathbf{1}$ such that $(v, \overline{v})$ is a bipartition of R.

Idea, following the proof of the "remaining direction":

- Compute a splitting S of R^*.
- Compute a mapping F such that $F \subseteq S^{\mathsf{T}}$.
- Compute $v := (R;R)^*;F^{\mathsf{T}};\mathsf{L}$.

Relational program for **computing a splitting** of an equivalence relation:

$$
\begin{array}{l}
\{\, \mathsf{I} \subseteq P \wedge P = P^{\mathsf{T}} \wedge P;P \subseteq P \,\} \\
w := \mathit{point}(P;\mathsf{L}); \\
\textbf{while } P;w \neq P;\mathsf{L} \textbf{ do} \\
\quad w := w \cup \mathit{point}(\overline{P;w} \cap P;\mathsf{L}) \textbf{ od} \\
\{\, w = w;\mathsf{L} \wedge P \cap w;w^{\mathsf{T}} \subseteq \mathsf{I} \wedge w;\mathsf{L} \subseteq P;\mathsf{L} \wedge P;w = P;\mathsf{L} \,\} \\
S := P;\mathit{inj}(w)^{\mathsf{T}} \\
\{\, S;S^{\mathsf{T}} = P \wedge S^{\mathsf{T}};S = \mathsf{I} \,\}
\end{array}
$$

Formal assertion-based verification by R.B. and M. Winter (Acta Informatica 47, pp. 77-110, 2010) using that $\mathit{point}(v)$ selects a point from a non-empty vector, axiomatised by,

$$
\mathit{point}(w);\mathsf{L} = \mathit{point}(w) \qquad \mathit{point}(w) \neq \mathsf{O} \qquad \mathit{point}(w);\mathit{point}(w)^{\mathsf{T}} \subseteq \mathsf{I},
$$

and $\mathit{inj}(w)$ is the **embedding mapping** generated by w, axiomatised by

$$
\mathit{inj}(w)^{\mathsf{T}};\mathit{inj}(w) \subseteq \mathsf{I} \qquad \mathit{inj}(w);\mathit{inj}(w)^{\mathsf{T}} = \mathsf{I} \qquad \mathit{inj}(w)^{\mathsf{T}};\mathsf{L} = w.
$$

The proof outline remains also correct if its relational program and the assertions are modified as follows:

$$\{\, \mathsf{I} \subseteq P \wedge P = P^\mathsf{T} \wedge P;P \subseteq P \,\}$$
$$w := \mathit{point}(P;\mathsf{L});$$
$$\textbf{while } P;w \neq P;\mathsf{L} \textbf{ do}$$
$$\quad w := w \cup \mathit{point}(\overline{P;w} \cap P;\mathsf{L}) \textbf{ od}$$
$$\{\, \mathsf{I} \subseteq P \wedge w = w;\mathsf{L} \wedge P \cap w;w^\mathsf{T} \subseteq \mathsf{I} \wedge w;\mathsf{L} \subseteq P;\mathsf{L} \wedge P;w = P;\mathsf{L} \,\}$$
$$\cancel{S := P;\mathit{inj}(w)^\mathsf{T}}$$
$$\{\, \mathsf{I} \subseteq P \wedge \exists S : S = P;\mathit{inj}(w)^\mathsf{T} \wedge S;S^\mathsf{T} = P \wedge S^\mathsf{T};S = \mathsf{I} \,\}$$

The new post-condition implies that there exists a splitting S of P with

$$inj(w) = inj(w);\mathsf{I} \subseteq inj(w);P^{\mathsf{T}} = (P;inj(w)^{\mathsf{T}})^{\mathsf{T}} = S^{\mathsf{T}}.$$

The axioms of $inj(w)$ say that $inj(w)$ is a mapping. So, we can take $inj(w)$ as F and get that F becomes superfluous by an axiom of $inj(w)$:

$$v := (R;R)^*;inj(w)^{\mathsf{T}};\mathsf{L} = (R;R)^*;w$$

Final program:

$$
\boxed{
\begin{array}{l}
\{\, R^* = (R^*)^{\mathsf{T}} \wedge R;(R;R)^* \cap \mathsf{I} = \mathsf{O} \,\} \\
P := R^*; \\
w := point(P;\mathsf{L}); \\
\textbf{while } P;w \neq P;\mathsf{L} \textbf{ do} \\
\quad w := w \cup point(\overline{P;w} \cap P;\mathsf{L}) \textbf{ od} \\
v := (R;R)^*;w \\
\{\, (v, \overline{v}) \text{ bipartion of } R \,\}
\end{array}
}
$$

RelView-function for testing a set-theoretic relation R to be bipartite:

```
isbipartite(R) =
  empty(R*refl(trans(R*R)) & I(R)).
```

RelView-program for computing for a set-theoretic bipartite relation R a vector v such that $(v, \overline{v})$ is a bipartition of R:

```
bipartition(R)
  DECL P, v, w
  BEG  P = refl(trans(R));
       w = point(dom(P));
       WHILE -eq(P*w,dom(P)) DO
         w = w | point(-(P*w) & dom(P)) OD;
       v = refl(trans(R*R)) * w
       RETURN v
  END.
```

Concluding Remarks

- The formality of algebraic proofs and their primary use of rewriting is a vantage point for the use of **tools for theorem proving**.
- Concerning this work, we startet with the automated theorem prover **Prover9**.
- Prover9 was not able to verify the more complex results without any user interaction.
- These restrictions became so serious that the change to a proof assistant was virtual essential.
- With the proof assistant **Coq** and the library "Relation algebra and KAT in Coq" from

  `http://perso.ens-lyon.fr/damien.pous/ra/`

  (author: D. Pous) we have verified all proofs of the paper.
- The proof scripts for all Coq proofs can be found in the web.

  `http://media.informatik.uni-kiel.de/Ramics2015/`