Generalised N-ary Relations and Allegories

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N-ary Relations and Allegories

- Allegories are a categorical generalisation of relation algebras where arrows are like binary relations.
- Ability to represent n-ary relations (for finite $n > 2$) is useful, e.g. for database applications.
- Given relational products, n-ary relations can be represented in allegory as binary relations between product objects.
- There are many ways of dividing the "legs" of an n-ary relation between source and target, vectorization (or its dual) seems the least arbitrary but, e.g., joins and intersections require different choices
- We develop a formalism for allegorical generalisations of n-ary relations based on (the dual of) vectorization and n-ary relational products, filling some technical details, particularly about properties of n-ary relational products.

Axioms of Allegories

- Intersection operators make each hom-set a meet semi-lattice.
  - We denote the associated partial order by $\sqsubseteq$, i.e., $R \sqsubseteq S :\equiv R \sqcap S = R$, for all $R, S \in \mathrm{Arr}_{\mathscr{A}}(A, B)$.
- In addition $\cdot \sqcap \cdot$ and $(\cdot)^\circ$ have to satisfy the following formulas, for all $R, S, T \in \mathrm{Arr}[\mathscr{A}]$ such that the formulas are well defined:

$$R^{\circ\circ} = R, \quad (RS)^\circ = S^\circ R^\circ, \quad (R \sqcap S)^\circ = R^\circ \sqcap S^\circ,$$
$$R(S \sqcap T) \sqsubseteq RS \sqcap RT, \quad RS \sqcap T \sqsubseteq (R \sqcap TS^\circ)S.$$

Examples of allegories

$\mathscr{R}$ (resp. $\mathscr{R}[\Lambda]$) where objects are sets and arrows are binary relations (resp. Λ-valued binary relations).

Relational Products

Definition

A pair of arrows $\bullet \xleftarrow{\pi_1} C \xrightarrow{\pi_2} \bullet$ is called a relational product iff

$$\pi_1^\circ \pi_1 = \mathrm{id}_{\overrightarrow{\pi_1}}, \quad \pi_2^\circ \pi_2 = \mathrm{id}_{\overrightarrow{\pi_2}},$$
$$\pi_1 \pi_1^\circ \sqcap \pi_2 \pi_2^\circ = \mathrm{id}_C,$$
$$\pi_1^\circ \pi_2 = \top_{\overrightarrow{\pi_1}\overrightarrow{\pi_2}}$$

- The relational product is the categorical product in the subcategory of maps.
- It is determined only up to an isomorphism
- We will name the common source of π_i's as $\overrightarrow{\pi_1} \times \overrightarrow{\pi_2}$
- In $\mathscr{R}$ and $\mathscr{R}[\Lambda]$ relational product is isomorphic with the cartesian product.

Example: Unary Products and Units

Definition

An object 1 in allegory $\mathscr{A}$ is called a unit whenever $\mathrm{id}_1 = \top_{11}$ and for any $A \in \mathrm{Obj}[\mathscr{A}]$ there exists some $u_A \in \mathrm{Arr}_{\mathscr{A}}(A, 1)$ which is total. An allegory where a unit exists is called unitary. Units generalize singleton sets.

Lemma

For any $A \in \mathrm{Obj}[\mathscr{A}]$ an arrow u_A is a map and $u_A = \top_{A1}$. Moreover, for any $A, B \in \mathrm{Obj}[\mathscr{A}]$ we have $u_A; (u_B)^\circ = \top_{AB}$.

Lemma

Suppose that $\mathscr{A}$ is a unitary allegory. Then for any $A \in \mathrm{Obj}[\mathscr{A}]$ the pair $A \xleftarrow{\mathrm{id}_A} A \xrightarrow{u_A} 1$ is a relational product.

The unit in a unitary allegory can be viewed as a 0-ary relational product.

n-ARY RELATIONAL PRODUCTS

To get n-ary products for $n > 2$ one can iterate binary ones. Sometimes, a direct algebraic characterization of n-ary relational products becomes handy.

DEFINITION

A finite family of arrows $\{\pi_i\}_{i \in I} \subseteq \mathrm{Arr}[\mathscr{A}]$ with a common source C, t is called an n-ary relational product iff it satisfies the following conditions:

$$\forall i \in I \, . \, \pi_i^\circ \pi_i = \mathrm{id}_{\overrightarrow{\pi_i}},$$

$$\bigsqcap_i \pi_i \pi_i^\circ = \mathrm{id}_C,$$

$$\forall k \in I \, . \, \Big(\bigsqcap_{i \in I \setminus \{k\}} \pi_i \pi_i^\circ \Big) \pi_k = \top_{\overleftarrow{\pi_k}\,\overrightarrow{\pi_k}}$$

Binary relational products are 2-ary relational products and vice versa.

n-ARY RELATIONAL PRODUCTS AND FACTORIZATION

n-ary relational products are categorical products in the subcategory of maps.

LEMMA

if $\{\pi_i\}_{i\in I}$ is an n-ary relational product with a common source C and $\{\ A \xrightarrow{f_i} \overrightarrow{\pi_i}\ \}_{i\in I}$ is a family of maps then the unique map $A \xrightarrow{f} C$ such that $f\pi_i = f_i$ for all $i \in I$ is given by the formula

$$f = \prod_{i\in I} f_i \pi_i^\circ .$$

LEMMA

Let $\{\ C \xrightarrow{\pi_i} \bullet\ \}_{i\in I}$ be an n-ary relational product. For any $\varnothing \neq J \subsetneq I$ and a family of total arrows $\{\ A \xrightarrow{R_i} \overrightarrow{\pi_i}\ \}_{i\in J}$ the arrow $R_J := \prod_{i\in J} R_i \pi_i^\circ$ is total.

Sharpness

Suppose that $\{\pi_i\}_{i \in I}$ is an n-ary relational product, where $I = \{1, \ldots, n\}$. Let $\{R_i\}_{i \in I}$ and $\{S_i\}_{i \in I}$ be two families of arrows such that

$$
\begin{tikzcd}
 & & \bullet & & \\
\bullet \arrow[urr, "R_1"] \arrow[drr, "R_n"'] & \vdots & \bullet \arrow[u, "\pi_1"] \arrow[d, "\pi_n"'] & \vdots & \bullet \arrow[ull, "S_1"'] \arrow[dll, "S_n"] \\
 & & \bullet & &
\end{tikzcd}
$$

We would like the factorisation through the relational product to commute with the composition of R_i's with S_i°'s, i.e., to satisfy the **sharpness condition**:

$$\left(\prod_{i \in I} R_i \pi_i^\circ\right)\left(\prod_{j \in I} \pi_j S_j^\circ\right) = \prod_{i \in I} R_i S_i^\circ .$$

Unfortunately sharpness condition is not satisfied in general allegories.

Generalization of Sharpness

The sharpness condition does not seem sufficient to prove the results we need. Therefore we introduce a more general condition:

Definition

Let $\{\pi_i\}_{i\in I}$ be a n-ary relational product and let $\{R_i\}_{i\in I}$ and $\{S_i\}_{i\in I}$ be families of arrows as in the definition of sharpness. We say that $\{\pi_i\}_{i\in I}$ satisfies the **generalised sharpness condition** for families $\{R_i\}_{i\in I}$ and $\{S_i\}_{i\in I}$ iff, for all non-empty $I_1, I_2 \subseteq I$ we have

$$\left(\prod_{i\in I_1} R_i\pi_i^\circ\right)\left(\prod_{j\in I_2} \pi_j S_j^\circ\right) = \left(\prod_{i\in I_1\cap I_2} R_i S_i^\circ\right).$$

An intersection of an empty family of arrows is a top arrow.

The generalised sharpness cannot be satisfied for arbitrary families of arrows (e.g., take disjoint I_1 and I_2 and consider R_i's and S_i's to be bottom arrows).

Applicability of the Generalization

Note that in general

$$\left(\prod_{i\in I_1} R_i\pi_i^\circ\right)\left(\prod_{j\in I_2} \pi_j S_j^\circ\right) \sqsubseteq \left(\prod_{i\in I_1\cap I_2} R_i S_i^\circ\right)$$

Indeed, the inequality obviously holds when $I_1 \cap I_2 = \varnothing$. If $I_1 \cap I_2 \neq \varnothing$ then for any $k \in I_1 \cap I_2$ we have $\left(\prod_{i\in I_1} R_i\pi_i^\circ\right)\left(\prod_{j\in I_2} \pi_j S_j^\circ\right) \sqsubseteq R_k S_k^\circ$, and thus $\left(\prod_{i\in I_1} R_i\pi_i^\circ\right)\left(\prod_{j\in I_2} \pi_j S_j^\circ\right) \sqsubseteq \prod_{k\in I_1\cap I_2} R_k S_k^\circ$. However,

Proposition

In $\mathscr{R}[\Lambda]$, for any locale Λ, the generalised sharpness condition is satisfied for arbitrary families of total arrows.

Unfortunately, the author does not know if the sharpness condition for total arrows implies generalised sharpness for total arrows.

Iterating Relational Products

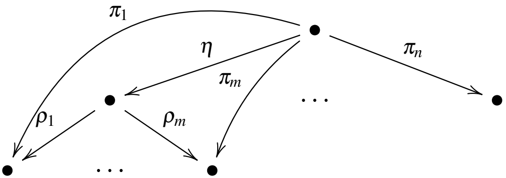
$$
\begin{tikzcd}
 & & \bullet \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_n"] & \\
 & \bullet \arrow[dl, "\rho_1"'] \arrow[dr, "\rho_m"] & \cdots & \bullet \\
\bullet & \cdots & \bullet &
\end{tikzcd}
$$

Is $\{\pi_1\rho_i\}_{1\leqslant i\leqslant m} \cup \{\pi_j\}_{2\leqslant j\leqslant n}$ an $(m+n-1)$-ary relational product if π_i's and ρ_i's are, respectively, n-ary and m-ary relational products?

Lemma

Suppose that $\{\, C \xrightarrow{\pi_i} \bullet \,\}_{i\in I}$ is an $|I|$-ary relational product, let $k \in I$ and let $\{\, \overrightarrow{\pi_k} \xrightarrow{\rho_j} \bullet \,\}_{j\in J}$ be a $|J|$-ary relational product. Suppose that π_i's and ρ_i's satisfy the generalised sharpness condition for total arrows. Then $\{\pi_k\rho_j\}_{j\in J} \cup \{\pi_i\}_{i\in I\setminus\{k\}}$ is an $(|I|+|J|-1)$-ary relational product also satisfying the generalised sharpness condition for total arrows.

DE-ITERATING RELATIONAL PRODUCTS



Assuming that π_i's and ρ_j's are relational products and η is the unique map such that $\pi_i = \eta\rho_i$, for all $1 \leqslant i \leqslant m$, is $\{\eta, \pi_{m+1}, \ldots, \pi_n\}$ a relational product?

LEMMA

Let $\{\ A \xrightarrow{\pi_i} \bullet\ \}_{i\in I}$ *and* $\{\ B \xrightarrow{\rho_j} \overrightarrow{\pi_j}\ \}_{j\in J}$*, where* $J \subsetneq I$*, be relational products. If* π_i*'s and* ρ_i*'s satisfy the generalised sharpness condition for maps, then* $\{\prod_{j\in J} \pi_j\rho_j^{\circ}\} \cup \{\pi_i\}_{i\in I\setminus J}$ *is an* $(|I| - |J| + 1)$*-ary relational product which satisfies the generalised sharpness condition for maps.*

Moving Legs Around

- In allegory, n-ary relations can be represented as arrows (binary relations) between products. This representation requires non-canonical division of the n "legs" of the relation into two groups.
- Different representations of the same n-ary relation are required so that some relational operations are applicable.
  - For instance, let R be a ternary relation with legs typed as A, B and C. Let $S \in \mathrm{Arr}(A,D)$ and $T \in \mathrm{Arr}(E,B)$ be binary relations. In order for S to be composable with R the latter should be represented as $R \in \mathrm{Arr}(B \times C, A)$. On the other hand, joinability of T with R requires $R \in \mathrm{Arr}(B, A \times C)$.
- Reciprocation allows to flip the legs of a binary relation.
- We still need, however, an operation which allows to move a leg of the n-ary relation from a source of representation to the target and vice-versa.
- We want to generalize to allegories the operation in $\mathscr{R}$ which relates $R \in \mathrm{Arr}_{\mathscr{R}}(A \times B, C)$ to $R' \in \mathrm{Arr}_{\mathscr{R}}(A, B \times C)$ where $aR'(b,c) \equiv (a,b)Rc$.
- Iteration and de-teration allows then manipulating more then 3 legs

Leg Moving Operation

We define the pair of maps $\mathrm{Arr}_{\mathscr{A}}(A \times B, C) \underset{\alpha^{A,B\times C}_{A\times B,C}}{\overset{\alpha^{A\times B,C}_{A,B\times C}}{\rightleftarrows}} \mathrm{Arr}_{\mathscr{A}}(A, B \times C)$ as

$$\alpha^{A\times B,C}_{A,B\times C}(R) := (\pi_1^{A\times B})^\circ ; \big(R;(\pi_2^{B\times C})^\circ \sqcap \pi_2^{A\times B};(\pi_1^{B\times C})^\circ\big),$$
$$\alpha^{A,B\times C}_{A\times B,C}(S) := \big(\pi_1^{A\times B};S \sqcap \pi_2^{A\times B};(\pi_1^{B\times C})^\circ\big);\pi_2^{B\times C}.$$

$$\begin{tikzcd}
 & & C & \\
A \arrow[rrr, dashed, bend right, "\alpha^{A\times B,C}_{A,B\times C}(R)"'] & A \times B \arrow[l, "\pi_1^{A\times B}"'] \arrow[ur, "R"] \arrow[dr, "\pi_2^{A\times B}"'] & & B \times C \arrow[ul, "\pi_2^{B\times C}"'] \arrow[dl, "\pi_1^{B\times C}"] \\
 & & B &
\end{tikzcd}$$

Note that $\alpha^{A,B\times C}_{A\times B,C}(S) = \big(\alpha^{C\times B,A}_{C,B\times A}(S^\circ)\big)^\circ$. One proves $\alpha^{A\times B,C}_{A,B\times C} = \left(\alpha^{A,B\times C}_{A\times B,C}\right)^{-1}$.

Relational Schemas

In relational algebra *á la* Codd "legs" (columns) of an n-ary relation are not ordered (not identified by position). Instead, they are identified by name. A set of column names of a given relation together with the assignement of a type to a column is called a schema of this relation. We mimic those ideas.

Definition

Let $\mathbb{T}$ be a fixed set of basic types (e.g., `integer`, `varchar`, etc.). A relation schema (X, α) over $\mathbb{T}$ consists of a finite set X of column names together with a mapping $\alpha : X \to \mathbb{T}$ assigning types to column names.

Definition

A category $\mathscr{S}[\mathbb{T}]$ of relation schemas over $\mathbb{T}$ has as objects relation schemas over $\mathbb{T}$. Morphisms $f : (X, \alpha) \to (Y, \beta)$ between relation schemas are injective maps $f : X \to Y$ between sets such that $\beta \circ f = \alpha$.

Note that $\mathscr{S}[\mathbb{T}]$ is a subcategory of a slice category $\mathrm{Set}/\mathbb{T}$.

Canonical Presentation of n-ary Relations

Definition (cf. vectorization of binary relations)

Let (X, α) be a relation schema over $\mathbb{T}$. Let $\mathscr{A}$ be a unitary allegory and let $[\![\cdot]\!] : \mathbb{T} \to \mathrm{Obj}[\mathscr{A}]$. A pair $(R, \{\, \overrightarrow{R} \xrightarrow{\pi_i} [\![\alpha(i)]\!] \,\}_{i \in X})$ is called **an instance of** (X, α) iff $\overleftarrow{R} = 1$ and $\{\pi_i\}_{i \in X}$ is an $|X|$-ary relational product.

- The representation depends on the choice of relational product.
- The set of column names is a part of the definition of an instance.
- Any bijection $f : (X, \alpha) \to (Y, \beta)$ gives rise to the renaming transformation $\hat{f}$ of instances (which corresponds to the renaming operation in the Codd's relational algebra):

$$\hat{f}((R, \{\, \overrightarrow{R} \xrightarrow{\pi_i} [\![\alpha(i)]\!] \,\}_{i \in X})) := (R, \{\, \overrightarrow{R} \xrightarrow{\pi_{f^{-1}(j)}} [\![\beta(j)]\!] \,\}_{j \in Y}).$$

Binary Relations Represented as 2-ary Relations

- Assume for simplicity that $\mathbb{T} = \mathrm{Obj}[\mathscr{A}]$ and that $[\![\cdot]\!] : \mathbb{T} \to \mathrm{Obj}[\mathscr{A}]$ is given by identity.
- Suppose that $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$ is a relational product of A and B.
- Observe that any arrow $R \in \mathrm{Arr}_{\mathscr{A}}(A, B)$ gives rise to an instance $(\alpha^{A,B}_{1,A\times B}(R), \{\pi_i\}_{i\in\{1,2\}})$ of a schema $(\{1,2\}, \{1 \mapsto A, 2 \mapsto B\})$.
- This assignement is invertible and thus we loose no information when we change into this representation.

Intersections of n-ary Relations

One can intersect only the instances of the same schemas with common product components. Let $(R, \{\pi_i\}_{i \in X})$ and $(S, \{\pi_i\}_{i \in X})$ be instances of the same relation schema (X, α). Then

$$(R, \{\pi_i\}_{i \in X}) \sqcap (S, \{\pi_i\}_{i \in X}) := (R \sqcap S, \{\pi_i\}_{i \in X}).$$

JOINS OF n-ARY RELATIONS

Suppose now that (X,α) and (Y,β) are relation schemas over $\mathbb{T}$ such that $X \cap Y$ is non-empty and $\alpha|_{X\cap Y} = \beta|_{X\cap Y}$. Moreover, let

$$\{\, A \xrightarrow{\rho_i^A} [\![\alpha(i)]\!] \,\}_{i\in X\setminus Y}, \quad \{\, B \xrightarrow{\rho_j^B} [\![\alpha(j)]\!] \,\}_{j\in X\cap Y}, \quad \{\, C \xrightarrow{\rho_k^C} [\![\beta(k)]\!] \,\}_{k\in Y\setminus X},$$

be relational products that satisfy the generalised sharpness property for maps. Consider instances $(R,\{\pi_i\}_{i\in X})$ of (X,α) and $(S,\{\sigma_i\}_{i\in Y})$ of (Y,β) such that π_i's and σ_i's satisfy the generalised sharpness condition for maps. Define

$$\pi_1^{A\times B} := \bigsqcap_{i\in X\setminus Y} \pi_i(\rho_i^A)^\circ, \quad \pi_2^{A\times B} := \bigsqcap_{i\in X\cap Y} \pi_i(\rho_i^B)^\circ,$$
$$\pi_1^{B\times C} := \bigsqcap_{i\in X\cap Y} \sigma_i(\rho_i^B)^\circ, \quad \pi_2^{B\times C} := \bigsqcap_{i\in Y\setminus X} \sigma_i(\rho_i^C)^\circ,$$

De-iterations $\{\pi_1^{A\times B}, \pi_2^{A\times B}\}$ and $\{\pi_1^{B\times C}, \pi_2^{B\times C}\}$ are relational products.

Using the deiterations we can now view R and S as 2-ary relations $R \in \mathrm{Arr}_{\mathscr{A}}(1, A \times B)$ and $S \in \mathrm{Arr}_{\mathscr{A}}(1, B \times C)$. When joining those two relations we need to preserve all the legs, thus we consider the following diagram:

$$\begin{tikzcd}
A \arrow[r, "\alpha^{1,A\times B}_{A,B}(R)"] & B \arrow[dr, "u_B"'] & & B \times C \arrow[ll, "\pi_1^{B\times C}"'] \\
& & 1 \arrow[ur, "S"'] &
\end{tikzcd}$$

An obvious map to construct is

$$\alpha^{1,A\times B}_{A,B}(R); \left(u_B; S \sqcap (\pi_1^{B\times C})^{\circ}\right) \in \mathrm{Arr}_{\mathscr{A}}(A, B \times C).$$

Moving the legs of this map around, deiterating and taking care of column names we obtain …

An Explicit Formula For Natural Join

Assume that $A \xleftarrow{\mu_1} D \xrightarrow{\mu_2} \overrightarrow{S}$ is also a relational product. Then we define a natural join of $(R, \{\pi_i\}_{i \in X})$ and $(S, \{\sigma_i\}_{i \in Y})$ as an instance $(R \bowtie S, \{\nu_i\}_{i \in X \cup Y})$ of a schema $(X \cup Y, \gamma)$ where

$$\gamma(i) := \begin{cases} \alpha(i) & \text{if } i \in X \\ \beta(i) & \text{if } i \in Y \end{cases}, \quad \nu_i := \begin{cases} \mu_1 \rho_i^A & \text{if } i \in X \setminus Y \\ \mu_2 \sigma_i & \text{if } i \in Y \end{cases},$$

$$R \bowtie S := \alpha_{1,A \times (B \times C)}^{A,B \times C}\big(\alpha_{A,B}^{1,A \times B}(R)(u_B S \sqcap (\pi_1^{B \times C})^\circ)\big) : 1 \to D.$$

It is easy to verify that in $\mathscr{R}$ this definition corresponds to the usual definition of a natural join of two relations in Codd's relational algebra.

CONCLUSION

- We presented a new approach to n-ary relations in allegories, proving on the way some interesting results on n-ary relational products.
- Our approach to n-ary relatiosn can be useful for database modeling.
- Considering n-ary relations in allegories allows us to transparently use Codd's relational algebra operations with various generalised relation-like constructs, (e.g., locale-valued relations).
- We would like to check if it is possible to develop similar results while working with weakened definitions of relational products.