Towards a Stochastic Interpretation of Game Logic

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Overview

EED.
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The formulas of a modal logic are given through the grammar

$$\varphi ::= p \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \Diamond\varphi$$

Here $p \in P$ is an atomic formula.
The usual conventions apply (e.g., $\varphi_1 \vee \varphi_2$ is written for $\neg(\neg\varphi_1 \wedge \neg\varphi_2)$, $\Box\varphi$ for $\neg\Diamond\neg\varphi$, etc).

Intuitively

We say that $\Diamond\varphi$ holds in state w iff we can make a transition from w to a state w' in which φ holds.

Kripke Model

A Kripke model (W, V, R) for the interpretation of the logic is given by

- a set W of states (or worlds),
- a map $V : P \to \mathbb{2}^W$, saying where the atomic formulas hold,
- a transition relation $R \subseteq W \times W$.

Validity

The validity relation $\models_K$ is defined recursively along the formula's structure:

- $w \models_K p$ iff $w \in V(p)$, whenever $p \in P$,
- $w \models_K \varphi_1 \wedge \varphi_2$ iff $w \models_K \varphi_1$ and $w \models_K \varphi_2$,
- $w \models_K \neg\varphi$ iff $w \models_K \varphi$ is false,
- $w \models_K \Diamond\varphi$ iff there exists w' with $\langle w, w' \rangle \in R$ such that $w' \models_K \varphi$.

This is what you would expect. One defines then morphisms of various kinds between models, has a look at expressivity (bisimilarity, logical and behavioral equivalence) through a relational or coalgebraic formulation, and in general have much fun with these models.

Neighborhoods

There is a more general way to interpret modal logics. For motivation, take a Kripke model (W, V, R), put $R(w) := \{w' \in W \mid \langle w, w' \rangle \in R\}$ as the set of successors to $w \in W$. Look at

$$M(w) := \{A \subseteq W \mid R(w) \cap A \neq \emptyset\}, \text{ (think of } \Diamond)$$
$$S(w) := \{A \subseteq W \mid R(w) \subseteq A\} \text{ (think of } \Box).$$

Thus, e.g., $A \in M(w)$ iff A contains some states which can be reached from w via R.

Note

$M(w)$ is an upper closed subset of $\mathcal{P}(W)$: $A \in M(w)$ and $A \subseteq B$ together imply $B \in M(w)$, similar for $S(w)$.

Neighborhood Model

A neighborhood model (W, V, N) ist defined just as a Kripke model, but N is a map from W to the upper closed subsets of $\mathbb{2}^W$.

$A \in N(w)$ means that A contains the states which can be reached by w.

Example

Each Kripke model generates neighborhood models.

Examples

Let $\mathcal{Z}(w)$ be the principal filter associated with w, i.e.,

$$A \in \mathcal{Z}(w) \Leftrightarrow w \in A.$$

Then $(W, V, \mathcal{Z})$ is a neighborhood model.
Given a topological space (W, τ), let $\mathcal{U}(w)$ be the neighbordhood filter of w with respect to τ. Then $(W, V, \mathcal{U})$ constitutes a neighborhood model.

Kripke models

A Kripke model is based on a coalgebra for the power set functor $\mathbb{2}^{-}$. This functor is the functorial part of the power set monad, thus Kripke models are based on Kleisli morphisms for that monad.

Neighborhood models

A neighborhood model is based on a coalgebra for the "upper closed" functor

$$Æ : W \mapsto \{A \subseteq W \mid A \text{ is upper closed}\}.$$

This functor is also the functorial part of a monad, thus neighborhood models are based on Kleisli morphisms for that monad.

Recall

A coalgebra (A, f) for a functor $\mathfrak{F}$ is an object A together with a morphism $f : A \to \mathfrak{F}(A)$.

Definition

Given a neighborhood model (W, V, N), define the validity sets $[\![\varphi]\!]$ for formulas φ inductively:

- $[\![p]\!] := V(p)$ for $p \in P$,
- $[\![\varphi_1 \wedge \varphi_2]\!] := [\![\varphi_1]\!] \cap [\![\varphi_2]\!]$,
- $[\![\neg\varphi]\!] := W \setminus [\![\varphi]\!]$,
- $[\![\Diamond\varphi]\!] := \{w \in W \mid [\![\varphi]\!] \in N(w)\}$.

Put $w \models_N \varphi$ iff $w \in [\![\varphi]\!]$.

Hence

Thus $w \models_N \Diamond\varphi$ iff $[\![\varphi]\!]$ contains the states which can be reached from w.

Observation

If the neighborhood model is generated from a Kripke model, then

$$w \models_K \varphi \Leftrightarrow w \models_N \varphi.$$

Thus neighborhood models are more general than Kripke models.

They are even strictly more general, since there certainly exist neighborhood models which are not generated from Kripke models.

Introduce actions into the logic

Let A be a set of actions; we introduce a family $(\langle a \rangle)_{a \in A}$ of modal operators. The idea is that formula $\langle a \rangle \varphi$ holds in a world w if action a leads to a world w' in which φ holds.

This is what the grammar now looks like:

$$\varphi ::= p \mid \varphi_1 \wedge \varphi_2 \mid \neg \varphi \mid \langle a \rangle \varphi$$

Here $p \in P$ is an atomic formula, and $a \in A$ is an action.

Modify models: Kripke

Associate with each action a a relation $R_a \subseteq W \times W$, and

$$w \models_K \langle a \rangle \varphi \Leftrightarrow R_a(w) \cap [\![\varphi]\!] \neq \emptyset.$$

Modify models: Neighborhood

Associate with each action a an upper closed subset $N_a \subseteq \mathbb{2}^W$, and

$$w \models_N \langle a \rangle \varphi \Leftrightarrow [\![\varphi]\!] \in N_a(w).$$

The set A of actions has been considered flat, i.e., without an inner structure. But sometimes the actions have a structure themselves. We look into

- Propositional Dynamic Logic (PDL)
- Game Logic

The actions are programs

A program π is built up from primitive programs:

- compose $\pi_1; \pi_2$: execute first π_1, then π_2,
- choose $\pi_1 \cup \pi_2$: decide whether to branch into π_1 or π_2,
- iterate π_1^*: execute program π_1 a finite number of times, including not at all,
- test $\varphi?$: check whether or not property φ holds.

E. g., $\varphi?; \pi_1 \cup (\neg\varphi)?; \pi_2$: If φ is satisfied, execute π_1, otherwise, execute π_2.

Grammar

Thus a program π is given by

$$\pi ::= t \mid \pi_1; \pi_2 \mid \pi_1 \cup \pi_2 \mid \pi^* \mid \varphi?$$

with $t \in \Pi$ a primitive program, and φ a formula of the underlying logic.

Interpretation?

A Kripke model for PDL is given through $(W, V, (R_t)_{t \in \Pi})$ with R_t a relation on W for each primitive program. Thus we want to piece together the relations R_π from the family $(R_t)_{t \in \Pi}$.

That's not too bad, given that each R_t is a Kleisli morphism for the power set monad, i.e., we have the Boolean operations, and a composition operator.

Here we go

Define recursively

$$R_{\pi_1 \cup \pi_2} := R_{\pi_1} \cup R_{\pi_2},$$
$$R_{\pi_1 ; \pi_2} := R_{\pi_1} \circ R_{\pi_2},$$
$$R_{\pi^*} := \bigcup_{n \geq 0} R_{\pi^n}$$

Test

Postpone the definition of $R_{\varphi ?}$ for the test operator. It depends on the semantics for the formulas.

Clearly

We have $w \models_K \langle \pi \rangle \varphi$ iff there exists $w' \in R_\pi(w)$ with $w' \models_K \varphi$.

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Transport this to neighborhood models. Let N map W to the upper closed subsets of $\mathbb{2}^W$, define

$$N'(A) := \{w \in W \mid A \in N(w)\}$$

for $A \subseteq W$.

Then $N' : \mathbb{2}^W \to \mathbb{2}^W$ is monotone. In fact: if $A \subseteq B$ and $A \in N(w)$, then $B \in N(w)$, since $N(w)$ is upper closed, hence $N'(A) \subseteq N'(B)$.

Thus

We can use the same machinery for neighborhood models, e.g. $N'_{\pi_1;\pi_2} := N'_{\pi_1} \circ N'_{\pi_2}$.

Semantics

$$[\![\langle\pi\rangle\varphi]\!] := N'_\pi([\![\varphi]\!]).$$

Then

One can show $w \models_K \varphi \Leftrightarrow w \models_N \varphi$, if the neighborhood model is generated from a Kripke model.
Thus the transition to neighborhood models is probably not worth the effort.
Is it?

But

The picture changes once we have a look at Game Logics.

Introducing Angel and Demon

The game is played between Angel and Demon. They move in turn.

Assume we are in world w, and Angel plays game γ. We want to know what this may achieve. Hence we want to know which worlds Angel can reach by playing γ in w.

This is certainly an upper closed set of subsets of W: If Angel can reach a world in A, and $A \subseteq B$, then Angel can reach a state in B as well.

Thus

Hence we want to assign to each game γ and each state $w \in W$ an upper closed subset $N'_\gamma(w) \subseteq \mathbb{2}^W$

Questions

Are Angel and Demon going to win something? What is a strategy? Let us briefly look at Banach-Mazur games.

Banach-Mazur

A B-M game is played on $\mathbb{N}$. Angel plays n_0. Depending on n_0, Demon plays n_1. Angel takes $\langle n_0, n_1\rangle$ into account and plays n_2, Demon reflects on $\langle n_0, n_1, n_2\rangle$ and counters with n_3, etc; the game never ends.
The actions are described through a trajectory in $\mathbb{N}^\infty$.

Strategy

This permits defining strategies as maps $\bigcup_{n\geq 0} \mathbb{N}^{2n} \to \mathbb{N}$ and $\bigcup_{n\geq 0} \mathbb{N}^{2n+1} \to \mathbb{N}$ for Angel resp. Demon.

WINNING STRATEGIES

The goal of the B-M game G_A is given by a subset $A \subseteq \mathbb{N}^\infty$. A strategy σ for Angel is a winning strategy for Angel iff, no matter what Demon does, the trajectory of the game is in A, when Angel plays according to σ. Similarly for Demon (all trajectories must then be in $\mathbb{N}^\infty \setminus A$).

DETERMINED GAMES

B-M game G_A is called determined iff either Angel or Demon has a winning strategy.

AXIOM OF DETERMINACY

Each B-M game G_A is determined.

Looks a bit far fetched ...

We know

The Axiom of Determinacy is incompatible with the Axiom of Choice.

Axiom of Choice

There exists a non-measurable subset of $[0,1]$.

Axiom of Determinacy

Every subset of $[0,1]$ is measurable.

What would Hamlet do?

The Axiom of Choice seems to be somewhat indispensable, but, on the other hand, the Axiom of Determinacy is also somewhat practical.

Returning to Game Logics

The modalities in Game Logics are games.

Games

Games are built up from primitive games in this way:

$$\gamma ::= g \mid \gamma^d \mid \gamma_1 ; \gamma_2 \mid \gamma_1 \cup \gamma_2 \mid \gamma_1 \cap \gamma_2 \mid \gamma^* \mid \gamma^\times \mid \varphi?$$

Here $g \in \Gamma$ is a primitive game, and φ a formula of the underlying logic.

Composition, $\gamma_1 \cup \gamma_2$ and γ^* are as in PDL, $\gamma_1 \cap \gamma_2$ is demonic choice, $\gamma^\times$ is demonic iteration. γ^d is demonization: Angel and Demon changes places.

PDL

Propositional Dynamic Logics is a fragment of Game Logics (just don't use the operators associated with Demon).

Angel and Demon play against each other. If the objective for Angel is to achieve a state in which a formula φ holds, the objective for Demon is to achive a state in which $\neg\varphi$ is true.

Strategy?

It is assumed that Angel and Demon follow some strategy. We do not say formally, however, what a strategy is (in contrast to B-M games), but rest on an informal understanding.

Determinedness

We assume that the game is determined in this sense: If Angel does not have a strategy for achieving a formula φ, Demon has a strategy for achieving $\neg\varphi$, and vice versa.

Some Consequences

It is basically sufficient to model Angel's behavior: $\gamma_1 \cap \gamma_2$ is equivalent to $(\gamma_1^d \cup \gamma_2^d)^d$, and $\gamma^\times$ is equivalent to $(\gamma^d)^{*d}$.

Neighborhood Model

The interpretation of game logics will be done through a neighborhood model $(W, V, (N_g)_{g \in \Gamma})$. Thus we have for each primitive game $g \in G$ and any world $w \in W$ an upper closed subset $N_g(w) \subseteq \mathbb{2}^W$.

Transform

Put $N_g'(A) := \{w \in W \mid A \in N_g(w)\}$. Then $N_g'(A)$ is the set of states which Angel can achieve when it plays primitive game $g \in \Gamma$ in a state taken from A.

It is convenient to assign to each game γ a map $N'_\gamma : \mathbb{2}^W \to \mathbb{2}^W$, as we did in the model case PDL.
We basically know from PDL how Angel operates by transporting the algebraic operations of programs to maps $\mathbb{2}^W \to \mathbb{2}^W$.

Demonization

Define for $N : \mathbb{2}^W \to \mathbb{2}^W$ its demonization

$$\partial N : A \mapsto W \setminus N(W \setminus A).$$

If $N'_\gamma : \mathbb{2}^W \to \mathbb{2}^W$ is defined for game γ, define $N'_{\gamma^d} := \partial N'_\gamma$.

Interpretation

We define in this manner inductively a map $N_\gamma : \mathbb{2}^W \to \mathbb{2}^W$ for each game γ. Then we set

$$[\![\langle\gamma\rangle\varphi]\!] := N'_\gamma([\![\varphi]\!]).$$

But ...

This is formulated for neighborhood models. Why don't we take Kripke models for interpreting game logics?

Each Kripke model spawns a neighborhood model, so Kripke models are not excluded.

There is a catch, though

It can be shown that games are distributive under the class of Kripke models, i.e.,

$$w \models_K \langle \gamma; (\gamma_1 \cup \gamma_2) \rangle \varphi \Longleftrightarrow w \models_K \big(\langle \gamma; \gamma_1 \rangle \cup \langle \gamma; \gamma_2 \rangle\big)\varphi$$

holds.

This is clearly inadequate.

Changes ahead

We want to say in a probablistic setting that a formula holds with a probability of at most 70% after performing some action. Hence our modal logics needs to be modified.

Modal logics

Formulas now look like this

$$\varphi ::= p \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \langle a :: r \rangle \varphi$$

with — again — $p \in P$ an atomic formula, $a \in A$ an action, and $r \in [0, 1]$ as a kind of threshold value.

The informal understanding is that formula $\langle a :: r \rangle \varphi$ holds in a world w iff the probability for transition from w after action a to a state in which φ holds is at least r.

INTERPRETATION?

The interpretation of this logic is done for general probabilities. Thus we do not stick to probability distributions over finite or countable sets.

RECAP

We assume that the set of worlds W is a measurable space. This is a set with a Boolean σ-algebra on it, the members are called events. A probability μ assigns to each event A its probability $\mu(A) \in [0, 1]$ with

- $\mu(\text{impossible event}) = \mu(\emptyset) = 0$, $\mu(\text{certain event}) = \mu(W) = 1$,
- $\mu(A \cup B) = \mu(A) + \mu(B)$, provided the events A and B are disjoint,
- if $A_1 \subseteq A_2 \subseteq \ldots$, then $\mu(A_1) \leq \mu(A_2) \leq \ldots$ and $\mu(\bigcup_{n\geq 1} A_n) = \sup_{n\in\mathbb{N}} \mu(A_n)$.

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Actually

The set $\$\,(W)$ of all probabilities on W is a measurable space itself.

But we know more

$ is an endofunctor on the category of all measurable spaces. It is the functorial part of a monad (which is sometimes called the Giry monad).

But we know still more

A stochastic relation K on W assigns to each $w \in W$ a probability $K(w) \in \$\,(W)$. $K(w)(A)$ is interpreted as the probability for a transition starting from w hitting an element of event A.
These are exactly the Kleisli morphisms for the Giry monad.

Stochastic relations are also known as Markov kernels or transition probabilities.

Model

We interpret this kind of modal logic through a stochastic Kripke model $(W, V, (K_a)_{a \in A})$. Now V maps the atomic formulas to the measurable sets (thus $V(p)$ becomes an event), and K_a is a stochastic relation on W for each $a \in A$.
$K_a(w)(A)$ is the probability that an a-transition from w ends up in an element of A.

Interpretation

The interpretation of the Boolean cases remains as it is, and

$$w \models_K \langle a :: r \rangle \varphi \iff K_a(w)([\![\varphi]\!]) \geq r.$$

Thus $\langle a :: r \rangle \varphi$ holds in w iff we can make an a-transition from w into a state in which φ holds with probability at least $r \in [0, 1]$.

Remark

Actually, we can do without negation in the logic, since

$$w \models_K \neg\varphi \text{ iff } w \notin [\![\varphi]\!].$$

$\mathbb{2}^{-}$

Power set monad.

$\mathbb{E}$

Upper closed subsets.

$

Giry monad.

Kleisli

Relations.

Kleisli

Neighborhoods.

Kleisli

Stochastic relations.

Logic

General modal logic.

Logic

General modal logic, game logic

Logic

Stochastic modal logic.

Uniform picture

The model is based in each case on a family of Kleisli morphisms for the corresponding monad. This leads to coalgebraic logic.

Requirements

A stochastic interpretation of a dynamic logic will have to deal with probabilities over the set of worlds (rather than the set of worlds proper).

state	$\Longrightarrow$	distribution over states
achievable states	$\Longrightarrow$	achievable events of state probabilities

Basic construction

$P_\gamma(w)$ is an upper closed subset of events over probability distributions on W for each $w \in W$. Then $G \in P_\gamma(w)$ means: Angel has a strategy for achieving a probability in G as a distribution for the next state, if it plays game γ in state w.

In particular

If $\{\mu \in \$(W) \mid \mu(A) \geq r\} \in P_\gamma(w)$, then the next state will be a member of event A with probability not less than r.

Q:

Can't we compose, then, the upper closed monad with the Giry monad?

A:

No, we can't.

Why?

The composition of two monads is not necessarily a monad again (You need a lot of machinery, i.e., natural transformations, to connect them). That's too bad.

Now, what?

We basically have to simulate the properties of a monad through a suitable functor. The composition of coalgebra morphisms is particularly important.

Parental guidance suggested

This may be suitable for mature audiences only, since there may be complications of a technical nature. I'll give you the Disney version, all cleaned up, and a bit distorted.

Take a model $\big(W, V, (P_g)_{g\in\Gamma}\big)$, where $P_g(w)$ is an upper closed subset of events over $\$\,(W)$ for simple games g.

Angelic choice for simple games

$$w \models \langle(g_1 \cup g_2) :: r\rangle\varphi \Leftrightarrow \begin{cases} \big(\{\mu \mid \mu([\![\varphi]\!]) \geq a_1\} \in P_{g_1}(w), \\ \text{or} \\ \{\mu \mid \mu([\![\varphi]\!]) \geq a_2\} \in P_{g_2}(w)\big), \\ \ \text{for all rational } a_1, a_2 \text{ with } a_1 + a_2 < r. \end{cases}$$

for $g_1, g_2 \in \Gamma$.

Demonization

Let $g \in \Gamma$, then $w \models \langle g^d :: r\rangle\varphi$ iff $\{\mu \mid \mu([\![\varphi]\!]) \geq r\} \in \partial P_g([\![\varphi]\!])$.

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COMPOSITION

Composition of two games is crucial (actually, it is exactly here that the composition of Kleisli morphisms would be very helpful).
We want to define $w \models \langle (g_1; g_2) :: r \rangle \varphi$ for $g_1, g_2 \in \Gamma$.

INTERMEDIARY STEP

Let

$$\mu \in H_{g_2}([\![\varphi]\!], q) \iff \int_0^1 \mu(\{w \mid w \models \langle g_2 :: r \rangle \varphi\})\, dr \geq q.$$

Then $\mu \in H_{g_2}([\![\varphi]\!], q)$ iff we can expect for distribution μ a g_2-transition to lead into $[\![\varphi]\!]$ with probability at least q. It is these guys we should be looking for.

COMPOSITION

$$w \models \langle (g_1; g_2) :: r \rangle \varphi \Leftrightarrow H_{g_2}([\![\varphi]\!], r) \in P_{g_1}(w).$$

Induction over the syntactical structure

In this way, we construct step by step the set

$$[\![\langle \gamma :: r \rangle \varphi]\!]$$

as the set of all states in which this formula holds, for all games γ and all formulas φ.

Theorem

$[\![\langle \gamma :: r \rangle \varphi]\!]$ is an event, provided the measurable space is complete.

The semantics of iteration, i.e. for $\langle \gamma^* :: r \rangle \varphi$, requires Boolean operations over an uncountable set, Boolean σ-algebras are usually only closed under countable operations. These *-operations are, however, well-behaved, and complete measurable spaces are closed under them (Souslin closure).

Special case: Kripke model

Each stochastic relation K on W gives an upper closed subset $P_K(w)$ of events over W for each world w:

$$P_K(w) := \{A \subseteq \$(W) \mid K(w) \in A\}.$$

Thus each Kripke model yields a stochastic model for Game Logic.

Theorem (cp. D. Kozen)

In a Kripke generated model, the semantics for the PDL fragment is the same as the semantics for PDL through the Kripke model.

Remark

Game Logic is distributive in a Kripke generated model.

Stochastic Nondeterminism

This is a model for stochastic nondeterminism:

- nondeterministic choice of different possibilities,
- the objects to choose from are probability distributions.

We need to impose some additional structure for modelling the desired structures, in particular composition.

On the monadic level

Nondeterminism $+$ randomness $\not\Rightarrow$ stochastic nondeterminsm.
This is so since we cannot compose the corresponding monads to obtain another monad.

P. d'Argenio and P. Sánchez Terraf (MCS)

Hit measurability of maps into the power set of $\$(W)$. Interesting results for expressivity.

P. Sánchez Terraf and EED. (JLC)

Find bridges between these approaches.