1 *Pointfree* Relational Calculus

1.1 Eindhoven quantifier calculus

de Morgan:

$$\neg\langle\forall\, i \;:\; R:\; T\rangle \quad=\quad \langle\exists\, i \;:\; R:\; \neg T\rangle \tag{1}$$
$$\neg\langle\exists\, i \;:\; R:\; T\rangle \quad=\quad \langle\forall\, i \;:\; R:\; \neg T\rangle \tag{2}$$

Trading:

$$\langle\forall\, i \;:\; R \wedge S:\; T\rangle \quad=\quad \langle\forall\, i \;:\; R:\; S \Rightarrow T\rangle \tag{3}$$
$$\langle\exists\, i \;:\; R \wedge S:\; T\rangle \quad=\quad \langle\exists\, i \;:\; R:\; S \wedge T\rangle \tag{4}$$

Splitting:

$$\langle\forall\, j \;:\; R:\; \langle\forall\, k \;:\; S:\; T\rangle\rangle \quad=\quad \langle\forall\, k \;:\; \langle\exists\, j \;:\; R:\; S\rangle:\; T\rangle \tag{5}$$
$$\langle\exists\, j \;:\; R:\; \langle\exists\, k \;:\; S:\; T\rangle\rangle \quad=\quad \langle\exists\, k \;:\; \langle\exists\, j \;:\; R:\; S\rangle:\; T\rangle \tag{6}$$

One-point:

$$\langle\forall\, k \;:\; k = e:\; T\rangle \quad=\quad T[k := e] \tag{7}$$
$$\langle\exists\, k \;:\; k = e:\; T\rangle \quad=\quad T[k := e] \tag{8}$$

Nesting:

$$\langle\forall\, a, b \;:\; R \wedge S:\; T\rangle \quad=\quad \langle\forall\, a \;:\; R:\; \langle\forall\, b \;:\; S:\; T\rangle\rangle \tag{9}$$
$$\langle\exists\, a, b \;:\; R \wedge S:\; T\rangle \quad=\quad \langle\exists\, a \;:\; R:\; \langle\exists\, b \;:\; S:\; T\rangle\rangle \tag{10}$$

Empty range:

$$\langle\forall\, k \;:\; \textsc{False}:\; T\rangle \quad=\quad \textsc{True} \tag{11}$$
$$\langle\exists\, k \;:\; \textsc{False}:\; T\rangle \quad=\quad \textsc{False} \tag{12}$$

1.2 Relational equality

Cyclic inclusion:

$$R = S \quad\equiv\quad R \subseteq S \,\wedge\, S \subseteq R \tag{13}$$

Indirect equality:

$$R = S \quad\equiv\quad \langle\forall\, X \;::\; (X \subseteq R \equiv X \subseteq S)\rangle \tag{14}$$
$$\equiv\quad \langle\forall\, X \;::\; (R \subseteq X \equiv S \subseteq X)\rangle \tag{15}$$

Functional equality:

$$f \subseteq g \equiv f = g \equiv f \supseteq g \tag{16}$$

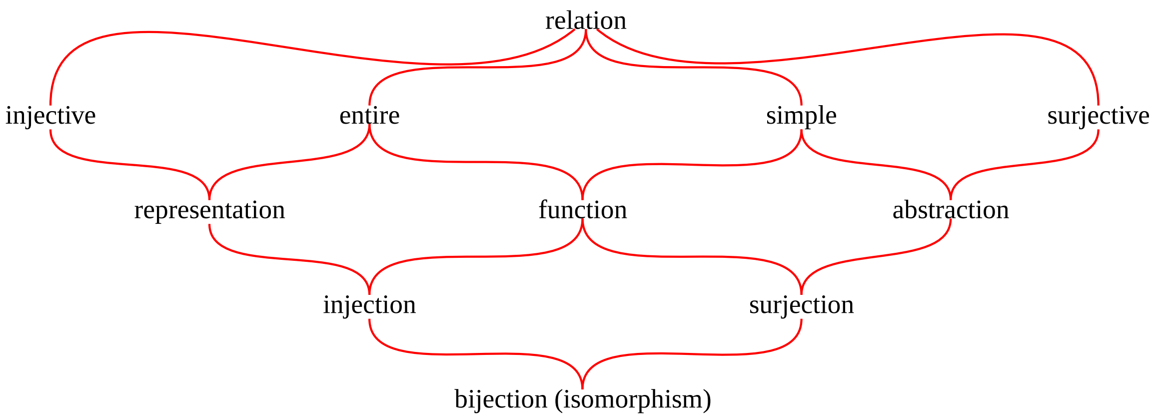
1.3 Relational taxonomy

Classification criteria:

	Reflexive	*Coreflexive*
$\mathsf{ker}\ R$	entire R	injective R
$\mathsf{img}\ R$	surjective R	simple R

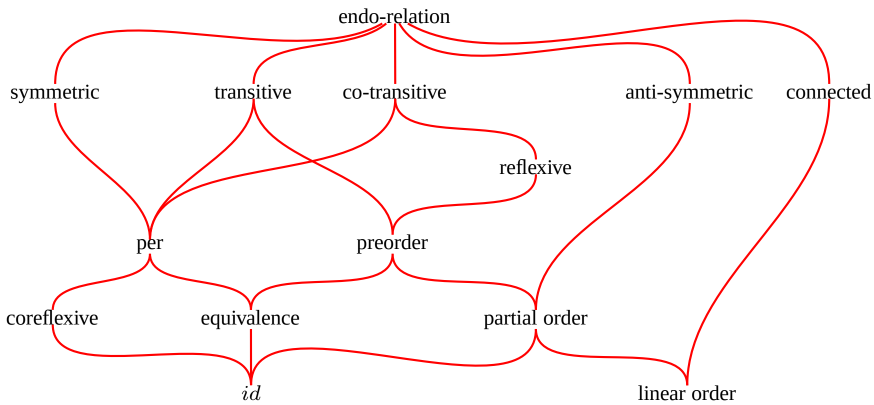
(17)

Binary relations:



(18)

Orders:



(19)

where

reflexive:	iff	$id_A \subseteq R$	(20)
coreflexive:	iff	$R \subseteq id_A$	(21)
transitive:	iff	$R \cdot R \subseteq R$	(22)
anti-symmetric:	iff	$R \cap R^\circ \subseteq id_A$	(23)
symmetric:	iff	$R \subseteq R^\circ (\equiv R = R^\circ)$	(24)
connected:	iff	$R \cup R^\circ = \top$	(25)

1.4 PF-transformation rules

“Guardanapo”:

$$b(f^\circ \cdot R \cdot g)a \quad \equiv \quad (f\ b)R(g\ a) \tag{26}$$

Left-division:

$$b\,(R \setminus Y)\,a \quad \equiv \quad \langle \forall\, c \;:\; c\,R\,b :\; c\,Y\,a \rangle \tag{27}$$

Pointwise ordering on functions:

$$f \mathrel{\dot{\sqsubseteq}} g \quad \equiv \quad f \subseteq \sqsubseteq \cdot\, g \quad \equiv \quad \langle \forall\, a \;::\; (f\ a) \sqsubseteq (g\ a) \rangle \tag{28}$$

1.5 Table of useful Galois connections

Relational Operators as Galois Connections

$$(f\ X) \subseteq Y \equiv X \subseteq (g\ Y)$$

Description	$f = g^\flat$	$g = f^\sharp$	**Obs.**
converse	$(\_)^\circ$	$(\_)^\circ$	
shunting rule	$(h\cdot)$	$(h^\circ\cdot)$	NB: h is a function
"converse" shunting rule	$(\cdot h^\circ)$	$(\cdot h)$	NB: h is a function
left-division	$(R\cdot)$	$(R \setminus\ )$	R under ...
right-division	$(\cdot R)$	$(\ / R)$	... over R
range	ρ	$(\cdot \top)$	lower $\subseteq$ restricted to coreflexives
domain	δ	$(\top \cdot)$	lower $\subseteq$ restricted to coreflexives
implication	$(R \cap\ )$	$(R \Rightarrow\ )$	Note that $(R \Rightarrow) = (\neg R \cup\ )$
difference	$(\_ - R)$	$(R \cup\ )$	
PROPERTIES			
cancellation	$X \subseteq (g \cdot f)X$	$(f \cdot g)Y \subseteq Y$	
definition	$f\ X = \bigcap\{Y \mid X \subseteq gY\}$	$g\ Y = \bigcup\{X \mid f\ X \subseteq Y\}$	
distribution	$f(X \cup Y) = (f\ X) \cup (f\ Y)$	$g(X \cap Y) = (g\ X) \cap (g\ Y)$	$f(\bigcup_i X_i) = \bigcup_i (f\ X_i)$ $g(\bigcap_i X_i) = \bigcap_i (g\ X_i)$

(29)

1.6 Other Galois connections

Meet-universal

$$X \subseteq (R \cap S) \quad \equiv \quad (X \subseteq R) \wedge (X \subseteq S) \tag{30}$$

Join-universal

$$(R \cup S) \subseteq X \quad \equiv \quad (R \subseteq X) \wedge (S \subseteq X) \tag{31}$$

Split-universal

$$X \subseteq \langle R, S \rangle \quad \equiv \quad \pi_1 \cdot X \subseteq R \ \wedge \ \pi_2 \cdot X \subseteq S \tag{32}$$

Either-universal

$$X = [R \, , S] \quad \equiv \quad X \cdot i_1 = R \quad \wedge \quad X \cdot i_2 = S \tag{33}$$

1.7 "Almost" Galois connections

"Shunting" rules for S a simple relation:

$$S \cdot R \subseteq T \quad \equiv \quad (\delta\, S) \cdot R \quad \subseteq \quad S^\circ \cdot T \tag{34}$$
$$R \cdot S^\circ \subseteq T \quad \equiv \quad R \cdot \delta\, S \quad \subseteq \quad T \cdot S \tag{35}$$

Variants concerning domain and range:

$$\delta\, R \subseteq X \quad \equiv \quad R \subseteq R \cdot X \tag{36}$$
$$\rho\, R \subseteq X \quad \equiv \quad R \subseteq X \cdot R \tag{37}$$

1.8 Converses

$$(R^\circ)^\circ \quad = \quad R \tag{38}$$
$$(R \cdot S)^\circ \quad = \quad S^\circ \cdot R^\circ \tag{39}$$

From the above:

$$\mathsf{ker}\,(R^\circ) \quad = \quad \mathsf{img}\, R \tag{40}$$
$$\mathsf{img}\,(R^\circ) \quad = \quad \mathsf{ker}\, R \tag{41}$$

1.9 Coreflexives

Coreflexives are symmetric and transitive:

$$\Phi^\circ = \Phi = \Phi \cdot \Phi \tag{42}$$

Meet of two coreflexives is composition:

$$\Phi \cap \Psi = \Phi \cdot \Psi \tag{43}$$

Since coreflexives are simple, the following follow from (34,35):

$$\Phi \cdot R \subseteq S \quad \equiv \quad \Phi \cdot R \subseteq \Phi \cdot S \tag{44}$$
$$R \cdot \Phi \subseteq S \quad \equiv \quad R \cdot \Phi \subseteq S \cdot \Phi \tag{45}$$

Mapping back and forward:

$$\Phi \subseteq \Psi \quad \equiv \quad \Phi \subseteq \top \cdot \Psi \tag{46}$$

Domain and range:

$$\begin{aligned} \delta\,(R \cap S) &= (R^\circ \cdot S) \cap id && (47)\\ \delta\,(R \cdot S) &= \delta\,(\delta\,R \cdot S) && (48)\\ \rho\,(R \cdot S) &= \rho\,(R \cdot \rho\,S) && (49)\\ \rho\,R &= \delta\,(R^\circ) && (50) \end{aligned}$$

Therefore, for Φ coreflexive

$$\delta\,\Phi = \Phi \tag{51}$$

Also a consequence of the above:

$$\begin{aligned} \delta\,R &= \mathsf{ker}\,R \cap id && (52)\\ \rho\,R &= \mathsf{img}\,R \cap id && (53) \end{aligned}$$

Other facts:

$$\begin{aligned} R &= R \cdot (\delta\,R) && (54)\\ R &= (\rho\,R) \cdot R && (55) \end{aligned}$$

Pre and post restriction:

$$\begin{aligned} R \cdot \Phi &= R \cap \top \cdot \Phi && (56)\\ \Psi \cdot R &= R \cap \Psi \cdot \top && (57) \end{aligned}$$

Domain/range elimination:

$$\begin{aligned} \top \cdot \delta\,R &= \top \cdot R && (58)\\ \rho\,R \cdot \top &= R \cdot \top && (59) \end{aligned}$$

1.10 Relational divisions

$$(R \setminus S) \cdot f = R \setminus (S \cdot f) \tag{60}$$

1.11 Meets

$$(S \cap T) \cdot R = (S \cdot R) \cap (T \cdot R) \quad \Leftarrow \quad T \cdot \mathsf{img}\,R \subseteq T \vee S \cdot \mathsf{img}\,R \subseteq S \tag{61}$$

Therefore, for f a function,

$$(S \cap T) \cdot f = (S \cdot f) \cap (T \cdot f) \tag{62}$$

$$R \cdot (S \cap T) = (R \cdot S) \cap (R \cdot T) \quad \Leftarrow \quad (\mathsf{ker}\,R) \cdot T \subseteq T \vee (\mathsf{ker}\,R) \cdot S \subseteq S \tag{63}$$

1.12 Splits

Definition equivalent to (32)

$$\langle R, S\rangle \quad = \quad \pi_1^\circ \cdot R \cap \pi_2^\circ \cdot S \tag{64}$$

The same definition pointwise: for all a, b, c

$$(a, b)\langle R, S\rangle c \quad \equiv \quad a\ R\ c\ \wedge\ b\ S\ c \tag{65}$$

Split cancellation

$$\pi_1 \cdot \langle R, S\rangle = R \cdot \delta\, S \quad \wedge \quad \pi_2 \cdot \langle R, S\rangle = S \cdot \delta\, R \tag{66}$$

Split (conditional) fusion [1]:

$$\langle R, S\rangle \cdot T = \langle R \cdot T, S \cdot T\rangle \quad \Leftarrow \quad R \cdot (\mathsf{img}\, T) \subseteq R \vee S \cdot (\mathsf{img}\, T) \subseteq S \tag{67}$$

Split absorption

$$\langle R \cdot T, S \cdot U\rangle \quad = \quad (R \times S) \cdot \langle T, U\rangle \tag{68}$$

Splits and converses:

$$\langle R, S\rangle^\circ \cdot \langle X, Y\rangle \quad = \quad (R^\circ \cdot X) \cap (S^\circ \cdot Y) \tag{69}$$

Therefore:

$$\mathsf{ker}\, \langle R, S\rangle \quad = \quad \mathsf{ker}\, R \cap \mathsf{ker}\, S \tag{70}$$

Product:

$$R \times S = \langle R \cdot \pi_1, S \cdot \pi_2\rangle \tag{71}$$

1.13 Eithers

Definition:

$$[R\ , S] \quad = \quad (R \cdot i_1^\circ) \cup (S \cdot i_2^\circ) \tag{72}$$

From (33), all coproduct properties extend to relations, in particular: $+$-reflexion:

$$id = [i_1\ , i_2] \tag{73}$$

etc. Eithers and converses:

$$[R\ , S] \cdot [T\ , U]^\circ \quad = \quad (R \cdot T^\circ) \cup (S \cdot U^\circ) \tag{74}$$

Coproducts:

$$R + S = [i_1 \cdot R\ , i_2 \cdot S] \tag{75}$$

1.14 Dedekind's rule

Also known as the **modular law**:

$$R \cdot S \cap T \quad \subseteq \quad R \cdot (S \cap R^\circ \cdot T) \tag{76}$$

[1] Theorem 12.30 in [1].

1.15 Relational projection

Definition

$$\pi_{g,f} R \quad \stackrel{\mathrm{def}}{=} \quad g \cdot R \cdot f^{\circ} \tag{77}$$

Property

$$\pi_{g,f} R \subseteq S \quad \equiv \quad g(S \leftarrow R)f \tag{78}$$

Wherever M is a simple, finite relation (coflexives included), and $projg, fM$ is also simple, this can be written pointwise as follows

$$\pi_{g,f} M \quad = \quad \{f\ a \mapsto g(M\ a) \mid a \in\ dom\ M\} \tag{79}$$

References

[1] C. Aarts, R.C. Backhouse, P. Hoogendijk, E.Voermans, and J. van der Woude. A relational theory of datatypes, December 1992. Available from `www.cs.nott.ac.uk/~rcb`.