1 Adenda aos sumários de MFP-II/0203

1.1 Known facts

Membership:

$$\in_{\mathcal{P}} \stackrel{\text{def}}{=} \in \tag{50}$$

$$\in_{C} \stackrel{\text{def}}{=} \perp \tag{51}$$

$$\in_{\lambda X.X} \stackrel{\text{def}}{=} id \tag{52}$$

$$\in_{\mathsf{F}\times\mathsf{G}} \stackrel{\text{def}}{=} (\in_{\mathsf{F}} \cdot \pi_1) \cup (\in_{\mathsf{G}} \cdot \pi_2) \tag{53}$$

$$\in_{\mathsf{F}+\mathsf{G}} \stackrel{\text{def}}{=} [\in_{\mathsf{F}}, \in_{\mathsf{G}}] \tag{54}$$

1.2 Proof of (10), p. 4

$$\begin{array}{ll}
 & [R,S] \subseteq X \\
\equiv & \qquad \{ \text{ property (11) } \} \\
 & R \subseteq X \cdot i_1 \ \wedge S \subseteq X \cdot i_2 \\
\equiv & \qquad \{ \text{ property (9) — a Galois connection (see p. 21)} \} \\
 & R \cdot i_1^\circ \subseteq X \ \wedge S \cdot i_2^\circ \subseteq X \\
\equiv & \qquad \{ \cup \text{ universal property} \} \\
 & (R \cdot i_1^\circ \cup S \cdot i_2^\circ) \subseteq X \\
ie: & \qquad \{ \text{ indirection} \} \\
 & [R,S] = R \cdot i_1^\circ \cup S \cdot i_2^\circ
\end{array}$$

1.3 Calculation of (13), p. 5

We have $\mathsf{F}\,X = A \times (1+X) \times (1+X)$ and thus $A \times (1+K) \times (1+K) \xleftarrow{\sigma} K$. Then

$$\begin{array}{lll}
\in_{\mathsf{F}} & = & \in_{A\times(1+X)\times(1+X)} \\
 & = & \qquad \{ \text{ (53) } \} \\
 & & \in_A \cdot\pi_1 \cup \in_{1+X} \cdot\pi_2 \cup \in_{1+X} \cdot\pi_3 \\
 & = & \qquad \{ \text{ (51) } \} \\
 & & \perp \cup [\perp, \in_X] \cdot \pi_2 \cup [\perp, \in_X] \cdot \pi_3 \\
 & = & \qquad \{ \ \perp \cup X = X \text{ and (52) } \} \\
 & & [\perp, id] \cdot \pi_2 \cup [\perp, id] \cdot \pi_3 \\
 & = & \qquad \{ \text{ (10), natural-}id \text{ and } \perp \cdot R = \perp \ \} \\
 & & i_2^\circ \cdot \pi_2 \cup i_2^\circ \cdot \pi_3
\end{array}$$

1.4 Calculation of (14), p. 5

$$\begin{array}{lll}
\prec_\sigma & = & \in_{\mathsf{F}} \cdot\sigma \\
 & = & \qquad \{ \text{ (13) } \} \\
 & & (i_2^\circ \cdot \pi_2 \cup i_2^\circ \cdot \pi_3) \cdot \sigma \\
 & = & \qquad \{ \text{ distributivity property of Galois connection lower adjoint } (\cdot\sigma) \text{ — see p. 21} \} \\
 & & i_2^\circ \cdot \pi_2 \cdot \sigma \cup i_2^\circ \cdot \pi_3 \cdot \sigma
\end{array}$$

$$
\begin{array}{cl}
= & \{ \text{ pointwise } \sigma \text{ as a finite mapping comprehension } \} \\
& i_2^\circ \cdot \pi_2 \cdot \{(\sigma\ k, k) \mid k \in \mathit{dom}\,\sigma\} \cup i_2^\circ \cdot \pi_3 \cdot \{(\sigma\ k, k) \mid k \in \mathit{dom}\,\sigma\} \\
= & \{ \text{ pointwise composition } \} \\
& \{(k', k) \mid k \in \mathit{dom}\,\sigma \wedge k'(i_2^\circ \cdot \pi_2)(\sigma\ k))\} \cup \{(k', k) \mid k \in \mathit{dom}\,\sigma \wedge k'(i_2^\circ \cdot \pi_3)(\sigma\ k))\} \\
= & \{ \text{ rule (8) } \} \\
& \{(k', k) \mid k \in \mathit{dom}\,\sigma \wedge (i_2\ k')(\pi_2)(\sigma\ k)\} \cup \{(k', k) \mid k \in \mathit{dom}\,\sigma \wedge (i_2\ k')(\pi_3)(\sigma\ k)\} \\
= & \{ \text{ VDM-SL model of } 1 + K \} \\
& \{(k', k) \mid k \in \mathit{dom}\,\sigma \wedge k' = \pi_2(\sigma\ k) \neq nil\} \cup \{(k', k) \mid k \in \mathit{dom}\,\sigma \wedge k' = \pi_3(\sigma\ k) \neq nil\} \\
= & \{ \text{ VDM-SL syntax } \} \\
& \{k \mapsto \sigma(k).mother \mid k \in\ set\ \mathit{dom}\,\sigma \wedge \sigma(k).mother <> nil\} \\
& \cup \\
& \{k \mapsto \sigma(k).father \mid k \in\ set\ \mathit{dom}\,\sigma \wedge \sigma(k).father <> nil\}
\end{array}
$$

1.5 Proof of (16), p. 6

$$
\begin{array}{cl}
& [R, S] \cdot i_1 \\
= & \{ \text{ (10) } \} \\
& ((R \cdot i_1^\circ) \cup (S \cdot i_2^\circ)) \cdot i_1 \\
= & \{ \text{ distributivity property of Galois connection lower adjoint } (\cdot i_1) \text{ — see p. 21} \} \\
& (R \cdot i_1^\circ \cdot i_1) \cup (S \cdot i_2^\circ \cdot i_1) \\
= & \{ \ \mathit{ker}\, i_1 = id \text{ and } i_2^\circ \cdot i_1 = \perp \} \\
& R \cup S \cdot \perp \\
= & \{ \text{ trivia } \} \\
& R
\end{array}
$$

(The proof of $[R, S] \cdot i_2 = S$ is similar.)

1.6 Calculation of (17), p. 6

We want to calculate

$$GenDia \xleftarrow{dt2gd} DecTree$$

as the abstraction function of law (15), that is

$$dt2gd = (\!| in_{\mu\mathsf{F}} \cdot g |\!)_{\mathsf{G}}$$

for

$$
\left\{
\begin{array}{l}
\mathsf{F}\ X = A \times (1 + X) \times (1 + X) \\
\mathsf{F}\ f = id \times (id + f) \times (id + f) \\
in_{\mu\mathsf{F}} = mk\_DecTree
\end{array}
\right.
$$

$$
\left\{
\begin{array}{l}
\mathsf{G}\ X = A \times (2 \rightharpoonup X) \\
\mathsf{G}\ f = id \times (id \rightharpoonup f) = id \times (f \cdot) \\
in_{\mu\mathsf{G}} = mk\_GenDia
\end{array}
\right.
$$

and isomorphism (18):

$$
\begin{array}{rcl}
g & = & \langle \pi_1, \pi_{12}, \pi_{22} \rangle \cdot (id \times \langle \lambda f.f\ 1, \lambda f.f\ 2 \rangle) \cdot (id \times tot) \\
& = & \langle \pi_1, \lambda\sigma.((tot\ \sigma)1, (tot\ \sigma)2) \rangle
\end{array}
$$

Let us abbreviate $in_{\mu\mathsf{F}} \cdot g$ to h, that is $dt2gd = (\!|h|\!)_{\mathsf{G}}$. Since $dt2gd$ is to be calculated as a G-cata, we have (by cancellation)

$$dt2gd \cdot in \quad = \quad h \cdot (id \times (dt2gd \cdot))$$

that is

$$dt2gd(mk\_DecTree(a, \sigma)) \quad = \quad h(a, dt2gd \cdot \sigma)$$

that is (VDM-SL):

```
dt2gd(mk_DecTree (a,s)) ==
     h(a,{ k |-> dt2gd(s(k)) | k in set dom s})
```

Since (VDM-SL syntax),

```
g(a,s) == (a,
           if 1 in set dom s then s(1) else nil,
           if 2 in set dom s then s(2) else nil)
```

one has ($h = in_{\mu\mathsf{F}} \cdot g$)

```
h(a,s) == mk_GenDia(a,
                    if 1 in set dom s then s(1) else nil,
                    if 2 in set dom s then s(2) else nil)
```

Altogether:

```
dt2gd(mk_DecTree(a,s)) ==
      mk_GenDia(a,
                if 1 in set dom s then dt2gd(s(1)) else nil,
                if 2 in set dom s then dt2gd(s(2)) else nil);
```

Note: by construction, $dt2gd$ is an isomorphism because so is g.

1.7 Calculation of (19), p. 7

The reasoning is as follows (fill in the ...):

$$
\begin{array}{rcl}
EquipDb & = & Equip \rightharpoonup (char^\star \times \mathbb{N} \times (Unit \rightharpoonup \mathbb{N})) \\
& \cong & \qquad \left\{ \text{ associativity: } \begin{array}{l} F_1 = id \rightharpoonup \ldots \\ r_1 = id \rightharpoonup \langle\langle \pi_1, \pi_2 \rangle, \pi_3 \rangle \end{array} \right\} \\
& & Equip \rightharpoonup ((char^\star \times \mathbb{N}) \times (Unit \rightharpoonup \mathbb{N})) \\
& \leq & \qquad \left\{ \begin{array}{l} F_2 = njoin \\ r_2 = unnjoin \end{array} \right\} \\
& & (Equip \rightharpoonup (char^\star \times \mathbb{N})) \times ((Equip \times Unit) \rightharpoonup \mathbb{N}) \\
& = & \qquad \{ \text{ data type definitions, } \texttt{token} \text{ parametrized by } K \} \\
& & (K \rightharpoonup (char^\star \times \mathbb{N})) \times ((K \times (K + K)) \rightharpoonup \mathbb{N}) \\
& \cong & \qquad \left\{ \begin{array}{l} F_3 = id \times ((id \times \ldots) \rightharpoonup id) \\ r_3 = id \times ((id \times \ldots) \rightharpoonup id) \end{array} \right\} \\
& & (K \rightharpoonup (char^\star \times \mathbb{N})) \times ((K \times (2 \times K)) \rightharpoonup \mathbb{N}) \\
& \leq & \qquad \left\{ \begin{array}{l} F_4 = mkf \cdot \ldots \\ r_4 = \ldots \cdot mkr \end{array} \right\} \\
& & 2^{K \times char^\star \times \mathbb{N}} \times 2^{K \times 2 \times K \times \mathbb{N}}
\end{array}
$$

1.8 Proof of (31), p. 8

$$IsPermutation \cdot cons \cdot (id \times IsPermutation)$$

$$
\begin{array}{cl}
= & \{ \text{ definition of } IsPermutation \ \} \\
 & (ker\ seq2bag) \cdot cons \cdot (id \times ker\ seq2bag) \\
= & \{ \text{ definition of } ker\ seq2bag \ \} \\
 & (seq2bag^{\circ} \cdot seq2bag) \cdot cons \cdot (id \times seq2bag^{\circ} \cdot seq2bag) \\
= & \{ \ (\cdot) \text{ is associative and } (\!|\_|\!)\text{-cancellation (21) } \} \\
 & seq2bag^{\circ} \cdot (bcons \cdot (id \times seq2bag)) \cdot (id \times seq2bag^{\circ} \cdot seq2bag) \\
= & \{ \text{ natural-}id\text{, } (\cdot) \text{ is associative and } \times\text{-functor} \} \\
 & seq2bag^{\circ} \cdot bcons \cdot (id \times seq2bag \cdot seq2bag^{\circ}) \cdot (id \times seq2bag) \\
= & \{ \ seq2bag \text{ is a surjective function } \} \\
 & seq2bag^{\circ} \cdot bcons \cdot (id \times id) \cdot (id \times seq2bag) \\
= & \{ \text{ natural-}id\text{, } \times\text{-functor-id and } (\!|\_|\!)\text{-cancellation (21)} \} \\
 & seq2bag^{\circ} \cdot seq2bag \cdot cons \\
= & \{ \text{ definition of } IsPermutation \ \} \\
 & isPermutation \cdot cons
\end{array}
$$

1.9 Proof of (35), p. 11

$$
\begin{array}{cl}
 & (\!\langle k \rangle\!) \cdot in = k \cdot \mathsf{F}\,\langle id, (\!\langle k \rangle\!) \rangle \\
= & \{ \ in \text{ and } out \text{ are isomorphisms } \} \\
 & (\!\langle k \rangle\!) = k \cdot \mathsf{F}\,\langle id, (\!\langle k \rangle\!) \rangle \cdot out \\
= & \{ \text{ natural-}id \text{ and } \times\text{-absorption } \} \\
 & (\!\langle k \rangle\!) = k \cdot \mathsf{F}\,((id \times (\!\langle k \rangle\!)) \cdot \langle id, id \rangle) \cdot out \\
= & \{ \ \mathsf{F}\text{-functor } \} \\
 & (\!\langle k \rangle\!) = k \cdot \mathsf{F}\,(id \times (\!\langle k \rangle\!)) \cdot \mathsf{F}\,\langle id, id \rangle \cdot out \\
= & \{ \text{ definition of } \mathsf{G} \ \} \\
 & (\!\langle k \rangle\!) = k \cdot (\mathsf{G}\,(\!\langle k \rangle\!)) \cdot \mathsf{F}\,\langle id, id \rangle \cdot out \\
= & \{ \ (\!\langle k \rangle\!) \text{ as a } \mathsf{G}\text{-hylo equation } \} \\
 & (\!\langle k \rangle\!) = [\![\ k, \mathsf{F}\,\langle id, id \rangle \cdot out\ ]\!] \\
= & \{ \text{ hylo factorization } \} \\
 & (\!\langle k \rangle\!) = (\!| k |\!) \cdot [\!(\mathsf{F}\,\langle id, id \rangle \cdot out)\!]
\end{array}
$$

1.10 Proof of (36), p. 11

Let $f = id$ in the mutual-recursion law:

$$
\begin{array}{cl}
 & \left\{ \begin{array}{l} id \cdot in = h \cdot \mathsf{F}\,\langle id, g \rangle \\ g \cdot in = k \cdot \mathsf{F}\,\langle id, g \rangle \end{array} \right. \equiv \langle id, g \rangle = (\!|\langle h, k \rangle|\!) \\
\equiv & \{ \text{ paramorphisms (34) } \} \\
 & \left\{ \begin{array}{l} id \cdot in = h \cdot \mathsf{F}\,\langle id, g \rangle \\ g = (\!\langle k \rangle\!) \end{array} \right. \equiv \langle id, g \rangle = (\!|\langle h, k \rangle|\!) \\
\Rightarrow & \{ \text{ let } h = in \cdot \mathsf{F}\,\pi_1 \ \} \\
 & \left\{ \begin{array}{l} id \cdot in = (in \cdot \mathsf{F}\,\pi_1) \cdot \mathsf{F}\,\langle id, g \rangle \\ g = (\!\langle k \rangle\!) \end{array} \right. \equiv \langle id, g \rangle = (\!|\langle in \cdot \mathsf{F}\,\pi_1, k \rangle|\!)
\end{array}
$$

$$
\begin{array}{ll}
\equiv & \{ \text{ substitution of } g \text{ and functor } \mathsf{F} \ \} \\
& in = in \equiv \langle id, [\!(k)\!] \rangle = (\!| \langle in \cdot \mathsf{F}\, \pi_1, k \rangle |\!) \\
\Rightarrow & \{ \text{ Leibnitz } \} \\
& \pi_2 \cdot \langle id, [\!(k)\!] \rangle = \pi_2 \cdot (\!| \langle in \cdot \mathsf{F}\, \pi_1, k \rangle |\!) \\
\equiv & \{ \text{ cancellation } \} \\
& [\!(k)\!] = \pi_2 \cdot (\!| \langle in \cdot \mathsf{F}\, \pi_1, k \rangle |\!)
\end{array}
$$

1.11 Proof of (44), p. 13

Let $X = id$ in (43):

$$
\begin{tikzcd}
C \arrow[dr, "R"'] & & C \arrow[ll, "id"'] \arrow[dl, "S"] \\
& B \arrow[u, phantom, "\subseteq"] &
\end{tikzcd}
\qquad id \subseteq R \setminus S \equiv R \cdot id \subseteq S
$$

$$
\begin{array}{ll}
\equiv & \{ \text{ natural-}id \text{ and definition of a reflexive relation } \} \\
& R \setminus S \text{ is reflexive } \equiv R \subseteq S
\end{array}
$$

1.12 Proof of (45), p. 13

$$
\begin{array}{ll}
& R \text{ is transitive} \\
\equiv & \{ \text{ definition of transitive } R \ \} \\
& R \cdot R \subseteq R \\
\equiv & \{ \ (43) \ \} \\
& R \subseteq R \setminus R
\end{array}
$$

1.13 Calculation of (46), p. 13

Diagram:

$$
\begin{tikzcd}
\mathcal{P}B \arrow[dr, "\in"'] & & \mathcal{P}B \arrow[ll, "\in \setminus \in"'] \arrow[dl, "\in"] \\
& B \arrow[u, phantom, "\subseteq"] &
\end{tikzcd}
$$

From (47):

$$a(\in \setminus \in)c \quad \equiv \quad (\forall b . b \in a \Rightarrow b \in c)$$

that is, $\in \setminus \in$ is nothing but set-theoretic inclusion ($\subseteq$).

1.14 Proof of (48), p. 13

Note that pre-$S \longrightarrow i_1 \cdot S \ , \ i_2 \cdot !$ is total. Diagram:

$$
\begin{tikzcd}
B & & A \arrow[ll, "S"'] \arrow[d, "id"] \\
B + 1 \arrow[u, "i_1^\circ"] & & A \arrow[ll, "\text{pre-}S \longrightarrow i_1 \cdot S \ , \ i_2 \cdot !"]
\end{tikzcd}
$$

Proof:

$$S \vdash i_1^\circ \cdot (\text{pre-}S \longrightarrow i_1 \cdot S\ ,\ i_2 \cdot !)$$

$\equiv$ { (32) }

$$i_1^\circ \cdot (\text{pre-}S \longrightarrow i_1 \cdot S\ ,\ i_2 \cdot !) \cdot (dom\ S) \subseteq S$$

$\equiv$ { (38) and $[\![\text{pre-}S]\!] = dom\ S$ }

$$i_1^\circ \cdot ((i_1 \cdot S) \cdot dom\ S \cup (i_2 \cdot ! \cdot (id - dom\ S))) \cdot (dom\ S) \subseteq S$$

$\equiv$ { distributivity property of Galois connection lower adjoint $(\cdot dom\ S)$ — see p. 21}

$$(i_1^\circ \cdot i_1 \cdot S \cdot (dom\ S) \cdot (dom\ S)) \cup (i_1^\circ \cdot i_2 \cdot ! \cdot (id - dom\ S) \cdot (dom\ S)) \subseteq S$$

$\equiv$ { $S \cdot (dom\ S) = S$; i_1 is entire and injective ; $id - X$ and X are each other complements }

$$id \cdot S \cup (i_1^\circ \cdot i_2 \cdot ! \cdot \bot) \subseteq S$$

$\equiv$ { $\bot \cdot X = \bot$ }

$$S \cup \bot \subseteq S$$

$\equiv$ { $\bot \cup X = X$ }

$$S \subseteq S$$

$\equiv$ { reflexivity}

T

1.15 Calculation of (49), p. 13

Diagram:

$$\begin{tikzcd}
B & A \rightharpoonup B \arrow[l, "Find\ a"'] \arrow[d, "elems^\circ \cdot mkr"] \\
B \arrow[u, "id"] & (A \times B)^\star \arrow[l, "Findl\ a"] \arrow[d, "id"] \\
B + 1 \arrow[u, "i_1^\circ"] & (A \times B)^\star \arrow[l, "find\ a"]
\end{tikzcd}$$

Calculation:

$$find\ a \;=\; (a \in) \cdot elems \cdot \pi_1{}^\star \longrightarrow i_1 \cdot (Findl\ a)\ ,\ i_2 \cdot !$$

$\equiv$ { introduce $a\ belongs = (a \in) \cdot elems$}

$$(a\ belongs) \cdot \pi_1{}^\star \longrightarrow i_1 \cdot (Findl\ a)\ ,\ i_2 \cdot !$$

$\equiv$ { definition of $belongs$}

$$(([\,] =) \longrightarrow \underline{F}\ ,\ ((a =) \cdot hd \longrightarrow \underline{T}\ ,\ (a\ belongs) \cdot tl)) \cdot \pi_1{}^\star \longrightarrow i_1 \cdot (Findl\ a)\ ,\ i_2 \cdot !$$

$\equiv$ { natural-hd, natural-tl}

$$(([\,] =) \longrightarrow \underline{F}\ ,\ ((a =) \cdot \pi_1 \cdot hd \longrightarrow \underline{T}\ ,\ (a\ belongs) \cdot \pi_1{}^\star \cdot tl)) \longrightarrow i_1 \cdot (Findl\ a)\ ,\ i_2 \cdot !$$

$\equiv$ { $[\![\underline{F}]\!] = \bot$, $[\![\underline{T}]\!] = id$, then properties (40) and (41) }

$$([\,] =) \longrightarrow i_2 \cdot !\ ,\ (a =) \cdot \pi_1 \cdot hd \longrightarrow i_1 \cdot (Findl\ a)\ ,\ (a\ belongs) \cdot \pi_1{}^\star \cdot tl \longrightarrow i_1 \cdot (Findl\ a)\ ,\ i_2 \cdot !$$

$\equiv$ { “then´ ´ and “else´ ´ clauses of definition of $Findl$}

$$([\,] =) \longrightarrow i_2 \cdot !\ ,\ (a =) \cdot \pi_1 \cdot hd \longrightarrow i_1 \cdot \pi_2 \cdot hd\ ,\ (a\ belongs) \cdot \pi_1{}^\star \cdot tl \longrightarrow i_1 \cdot (Findl\ a) \cdot tl\ ,\ i_2 \cdot ! \cdot tl$$

$\equiv$ { (42) for $S = tl$ etc}

$$([\,] =) \longrightarrow i_2 \cdot !\ ,\ (a =) \cdot \pi_1 \cdot hd \longrightarrow i_1 \cdot \pi_2 \cdot hd\ ,\ ((a\ belongs) \cdot \pi_1{}^\star \longrightarrow i_1 \cdot (Findl\ a)\ ,\ i_2 \cdot !) \cdot tl$$

$$
\begin{array}{ll}
\equiv & \{ \text{ definition of } find \text{ (49) } \} \\
& ([\,] =) \longrightarrow i_2 \cdot ! \ , \ (a =) \cdot \pi_1 \cdot hd \longrightarrow i_1 \cdot \pi_2 \cdot hd \ , \ (find\ a) \cdot tl \\
\equiv & \{ \text{ property } P \longrightarrow R \ , \ (Q \longrightarrow S \ , \ T) = (id - P - Q) \longrightarrow T \ , \ (P \longrightarrow R \ , \ S) \} \\
& (\lambda l. \neg([\,] = l \vee a = \pi_1(hd\, l))) \longrightarrow (find\ a) \cdot tl \ , \ ([\,] =) \longrightarrow i_2 \cdot ! \ , \ i_1 \cdot \pi_2 \cdot hd
\end{array}
$$

So we have obtained a tail-recursive definition of $find$ from which the proposed `while`-loop can be encoded.

Relational Operators as Galois Conections

$$(f\ X) \subseteq Y \equiv X \subseteq (g\ Y)$$

Description	$f = g^\flat$	$g = f^\sharp$	**Obs.**
converse	$(\_)^\circ$	$(\_)^\circ$	
left-division	$(\cdot R)$	$(\ / R)$	
right-division	$(R\cdot)$	$(R \setminus\ )$	
shunting rule	$(f\cdot)$	$(f^\circ\cdot)$	NB: f is a function
“converse” shunting rule	$(\cdot f^\circ)$	$(\cdot f)$	NB: f is a function
range	*rng*	$(\cdot\top)$	lower $\subseteq$ restricted to coreflexives
domain	*dom*	$(\top\cdot)$	lower $\subseteq$ restricted to coreflexives
implication	$(R \cap\ )$	$(R \Rightarrow\ )$	Note that $(R \Rightarrow) = (\neg R \cup\ )$
difference	$(\_ - R)$	$(R \cup\ )$	
definition	$f\ X = \bigcap\{Y \mid X \subseteq gY\}$	$g\ Y = \bigcup\{X \mid f\ X \subseteq Y\}$	
distributive property	$f(X \cup Y) = (f\ X) \cup (f\ Y)$	$g(X \cap Y) = (g\ X) \cap (g\ Y)$	