The Data Cube as a Typed Linear Algebra Operator

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J.N. Oliveira

INESC TEC & U.Minho

(H2020-732051: CloudDBAppliance)

H.D. Macedo

SW Eng Group @ U.Aharus

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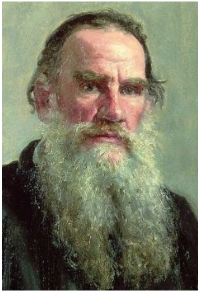
Motivation

"Only by taking infinitesimally small units for observation (the ***differential*** *of history, that is, the individual tendencies of men) and attaining to the art of* ***integrating*** *them (that is, finding the sum of these infinitesimals) can we hope to arrive at the laws of history."*

Leo Tolstoy, "War and Peace"
- Book XI, Chap.II (1869)



L. Tolstoy (1828–1910)

150 years later, this is what we are trying to attain through **data-mining**.

But — how fit are our **maths** for the task?

Have we attained the "**art of integration**"?

Motivation

Since the early days of psychometrics in the **social sciences** (1970s), **linear algebra** (LA) has been central to data analysis (e.g. tensor decompositions etc)

We follow this trend but in a **typed** way, merging **LA** with polymorphic **type systems**, over a categorial basis.

We address a concrete example: that of studying the maths behind a well-known device in data analysis, the **data cube** construction.

We will define this construction as a **polymorphic LA** operator.

Typing **linear algebra** is proposed as a strategy for achieving such an “**art of integration**”.

Running example

Raw data:

$t =$

#	**Model**	**Year**	**Color**	**Sale**
1	*Chevy*	1990	*Red*	5
2	*Chevy*	1990	*Blue*	87
3	*Ford*	1990	*Green*	64
4	*Ford*	1990	*Blue*	99
5	*Ford*	1991	*Red*	8
6	*Ford*	1991	*Blue*	7

Rows — records (n-many) — the *infinitesimals*
Columns — attributes — the *observables*

Column-orientation — each column (attribute) A represented by a function $t_A : n \to A$ such that $a = t_A\,(i)$ means "a is the value of attribute A in record nr i".

Records are tuples

Can records be rebuilt from such **attribute projection** functions?

Yes — by **tupling** them.

Tupling: *Given functions $f : A \to B$ and $g : A \to C$, their tupling is the function $f \triangledown g$ such that*

$$(f \triangledown g)\ a = (f\ a, g\ a)$$

For instance,

$$(t_{Color} \triangledown t_{Model})\ 2 = (Blue, Chevy),$$
$$(t_{Year} \triangledown (t_{Color} \triangledown t_{Model}))\ 3 = (1990, (Green, Ford))$$

and so on.

Inverting tuples

For the column-oriented model to work one will need to express *joins*, and these call for *"inverse"* functions, e.g.

$$(t_{Model} \triangledown t_{Year})^{\circ}\ (Ford, 1990) = \{3, 4\}$$

meaning that tuples nr 3 and nr 4 have the same model (*Ford*) and year (1990).

However, the type $f^{\circ} : A \to \mathcal{P}\ n$ is rather annoying, as it involves **sets** of tuple indices — these will add an extra layer of complexity.

Fortunately, there is a simpler way — **typed linear algebra**, also known as **linear algebra of programming** (**LAoP**).

The LAoP approach

Represent functions by Boolean matrices:

Given (finite) types A and B, any function

$$f : A \to B$$

can be represented by a matrix $[\![f]\!]$ with A-many columns and B-many rows such that, for any $b \in B$ and $a \in A$, the (b, a)-matrix-cell is

$$b\ [\![f]\!]\ a = \begin{cases} 1 \Leftarrow b = f\ a \\ 0\ \mathit{otherwise} \end{cases}$$

NB: Following the **infix** notation usually adopted for relations (which are Boolean matrices) — for instance $y \leqslant x$ — we write $y\ M\ x$ to denote the contents of the cell in matrix M addressed by row y and column x.

The LAoP approach

One projection function (matrix) per **dimension** attribute:

t_{Model}	1	2	3	4	5	6
Chevy	1	1	0	0	0	0
Ford	0	0	1	1	1	1

t_{Year}	1	2	3	4	5	6
1990	1	1	1	1	0	0
1991	0	0	0	0	1	1

t_{Color}	1	2	3	4	5	6
Blue	0	1	0	1	0	1
Green	0	0	1	0	0	0
Red	1	0	0	0	1	0

#	**Model**	**Year**	**Color**	**Sale**
1	Chevy	1990	Red	5
2	Chevy	1990	Blue	87
3	Ford	1990	Green	64
4	Ford	1990	Blue	99
5	Ford	1991	Red	8
6	Ford	1991	Blue	7

NB: we tend to abbreviate $[\![f]\!]$ by f when the context is clear.

The LAoP approach

Note how the inverse of a function is also represented by a Boolean matrix, e.g.

t°_{Model}	Chevy	Ford
1	1	0
2	1	0
3	0	1
4	0	1
5	0	1
6	0	1

versus

t_{Model}	1	2	3	4	5	6
Chevy	1	1	0	0	0	0
Ford	0	0	1	1	1	1

— no need for powersets.

Clearly,

$$j \; t^{\circ}_{Model} \; a = a \; t_{Model} \; j$$

Given a matrix M, M° is known as the **transposition** of M.

The LAoP approach

We **type** matrices in the same way as functions: $M : A \to B$ means a matrix M with A-many columns and B-many rows.

Matrices are arrows: $A \xrightarrow{M} B$ denotes a matrix from A (source) to B (target), where A, B are (finite) types.

Writing $B \xleftarrow{M} A$ means the same as $A \xrightarrow{M} B$.

Composition — *aka* matrix multiplication:

$$\begin{tikzcd} B & A \arrow[l, "M"'] & C \arrow[l, "N"'] \arrow[ll, bend left, "M\cdot N"] \end{tikzcd}$$

$$b(M \cdot N)c \;=\; \langle \sum a \;::\; (b\;M\;a) \times (a\;N\;c) \rangle$$

The LAoP approach

Function composition implemented by matrix multiplication, $[\![f \cdot g]\!] = [\![f]\!] \cdot [\![g]\!]$

Identity — the identity matrix *id* corresponds to the identity function and is such that

$$M \cdot id = M = id \cdot M \qquad (1)$$

Function tupling corresponds to the so-called **Khatri-Rao product** $M \triangledown N$ defined index-wise by

$$(b, c)\ (M \triangledown N)\ a \quad = \quad (b\ M\ a) \times (c\ N\ a) \qquad (2)$$

Khatri-Rao is a "column-wise" version of the well-known **Kronecker product** $M \otimes N$:

$$(y, x)\ (M \otimes N)\ (b, a) \quad = \quad (y\ M\ b) \times (x\ N\ a) \qquad (3)$$

Typing data

The raw data given above is represented in the LAoP by the expression

$$v = (t_{Year} \triangledown (t_{Color} \triangledown t_{Model})) \cdot (t^{Sale})^{\circ}$$

of type

$$v : 1 \to (Year \times (Color \times Model))$$

depicted aside.

Year x (Color x Model)			ALL
1990	Blue	Chevy	87
		Ford	99
	Green	Chevy	0
		Ford	64
	Red	Chevy	5
		Ford	0
1991	Blue	Chevy	0
		Ford	7
	Green	Chevy	0
		Ford	0
	Red	Chevy	0
		Ford	8

v is a **multi-dimensional** column vector — a **tensor**. Datatype $1 = \{\text{ALL}\}$ is the so-called **singleton** type.

Dimensions and measures

Sale is a special kind of data — a **measure**. Measures are encoded as **row** vectors, e.g.

t^{Sale}	1	2	3	4	5	6
1	5	87	64	99	8	7

recall

#	**Model**	**Year**	**Color**	**Sale**
1	Chevy	1990	Red	5
2	Chevy	1990	Blue	87
3	Ford	1990	Green	64
4	Ford	1990	Blue	99
5	Ford	1991	Red	8
6	Ford	1991	Blue	7

$$\begin{tikzcd} & Model & \\ Year & \#t \arrow[l, "t_{Year}"'] \arrow[u, "t_{Model}"'] \arrow[r, "t_{Color}"] \arrow[d, "t^{Sale}"] & Color \\ & 1 & \end{tikzcd}$$

Summary: **dimensions** *are* **matrices**, **measures** *are* **vectors**.

Measures provide for **integration** in Tolstoy's sense — *aka* **consolidation**

Totalisers

There is a unique function in type $A \to 1$, usually named $A \xrightarrow{!} 1$. This corresponds to a row vector wholly filled with 1s.

Example: $2 \xrightarrow{!} 1 = \begin{bmatrix} 1 & 1 \end{bmatrix}$

Given $M : B \to A$, the expression $! \cdot M$ (where $A \xrightarrow{!} 1$) is the row vector (of type $B \to 1$) that contains all column **totals** of M,

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 50 & 40 & 85 & 115 \\ 50 & 10 & 85 & 75 \end{bmatrix} = \begin{bmatrix} 100 & 50 & 170 & 190 \end{bmatrix}$$

Given type A, define its **totalizer** matrix $A \xrightarrow{\tau_A} A + 1$ by

$$\begin{aligned} \tau_A &: A \to A + 1 \\ \tau_A &= \begin{bmatrix} id \\ \hline ! \end{bmatrix} \end{aligned} \tag{5}$$

Thus $\tau_A \cdot M$ yields a copy of M on top of the corresponding totals.

Cubes

Data **cubes** are easily obtained from products of totalizers.

Recall the Kronecker (tensor) product $M \otimes N$ of two matrices:

$$\begin{array}{ccccc} A & & C & & A \times C \\ {\scriptstyle M}\downarrow & & {\scriptstyle N}\downarrow & & \downarrow{\scriptstyle M \otimes N} \\ B & & D & & B \times D \end{array}$$

The matrix

$$A \times B \xrightarrow{\tau_A \otimes \tau_B} (A+1) \times (B+1)$$

provides for totalization on the **two dimensions** A and B.

Indeed, type $(A+1) \times (B+1)$ is isomorphic to $A \times B + A + B + 1$, whose four parcels represent the four elements of the "**dimension powerset** of $\{A, B\}$".

Cube = muti-dimensional totalisation

Recalling

$$v = (t_{Year} \triangledown (t_{Color} \triangledown t_{Model})) \cdot (t^{Sale})^{\circ}$$

we build

$$c = (\tau_{Year} \otimes (\tau_{Color} \otimes \tau_{Model})) \cdot v$$

This is the multidimensional vector (tensor) representing the **data cube** for

- **dimensions** *Year*, *Color*, *Model*
- **measure** *Sale*

depicted aside.

(Year+1) x ((Color+1) x (Model+1))			ALL
1990	Blue	Chevy	87
		Ford	99
		ALL	186
	Green	Chevy	0
		Ford	64
		ALL	64
	Red	Chevy	5
		Ford	0
		ALL	5
	ALL	Chevy	92
		Ford	163
		ALL	255
1991	Blue	Chevy	0
		Ford	7
		ALL	7
	Green	Chevy	0
		Ford	0
		ALL	0
	Red	Chevy	0
		Ford	8
		ALL	8
	ALL	Chevy	0
		Ford	15
		ALL	15
ALL	Blue	Chevy	87
		Ford	106
		ALL	193
	Green	Chevy	0
		Ford	64
		ALL	64
	Red	Chevy	5
		Ford	8
		ALL	13
	ALL	Chevy	92
		Ford	178
		ALL	270

Totalisers yield cubes

Thanks to $\times$-**absorption**

$$(M \otimes N) \cdot (P \triangledown Q) = (M \cdot P) \triangledown (N \cdot Q) \tag{6}$$

we can simplify the definition:

$$\begin{array}{rl}
c & = \ (\tau_{Year} \otimes (\tau_{Color} \otimes \tau_{Model})) \cdot v \\
& = \qquad \{ \ v = (t_{Year} \triangledown (t_{Color} \triangledown t_{Model})) \cdot (t^{Sale})^{\circ} \ \} \\
& \quad (\tau_{Year} \otimes (\tau_{Color} \otimes \tau_{Model})) \cdot (t_{Year} \triangledown (t_{Color} \triangledown t_{Model})) \cdot (t^{Sale})^{\circ} \\
& = \qquad \{ \ \textbf{absorption}\text{-law } (6) \ \} \\
& \quad ((\tau_{Year} \cdot t_{Year}) \triangledown ((\tau_{Color} \cdot t_{Color}) \triangledown ((\tau_{Model} \cdot t_{Model})))) \cdot (t^{Sale})^{\circ} \\
& = \qquad \{ \ \text{define } t'_A = \tau_A \cdot t_A \ \} \\
& \quad (t'_{Year} \triangledown (t'_{Color} \triangledown t'_{Model})) \cdot (t^{Sale})^{\circ}
\end{array}$$

Note that $t'_A = \left[\frac{t_A}{!}\right]$, since t_A is a function.

Generalizing data cubes

In our approach a **cube** is not necessarily one such column vector.

The key to **generic** data cubes is (generalized) **vectorization**, a kind of "**matrix currying**": given $A \times B \xrightarrow{M} C$ with $A \times B$-many columns and C-many rows, reshape M into its **vectorized** version $B \xrightarrow{\mathbf{vec}_A\, M} A \times C$ with B-many columns and $A \times C$-many rows.

Such matrices, M and $\mathbf{vec}_A\, M$, are **isomorphic** in the sense that they contain the same information in different formats, cf

$$c\ M\ (a, b) \quad = \quad (a, c)\ (\mathbf{vec}_A\, M)\ b \tag{7}$$

which holds for every a, b, c.

Generalizing data cubes

Vectorization thus has an inverse operation — **unvectorization**:

$$
\begin{tikzcd}
A \times B \to C \arrow[rr, bend left, "\mathbf{vec}_A"] \arrow[rr, phantom, "\cong"] & & B \to A \times C \arrow[ll, bend left, "\mathbf{unvec}_A"]
\end{tikzcd}
$$

That is, M can be retrieved back from $\mathbf{vec}_A\, M$ by unvectorizing it:

$$N = \mathbf{vec}_A\, M \quad \Leftrightarrow \quad \mathbf{unvec}_A\, N = M \tag{8}$$

Vectorization has a rich algebra, e.g. a **fusion**-law

$$(\mathbf{vec}\, M) \cdot N \quad = \quad \mathbf{vec}\,(M \cdot (id \otimes N)) \tag{9}$$

and an **absorption**-law:

$$\mathbf{vec}\,(M \cdot N) \quad = \quad (id \otimes M) \cdot \mathbf{vec}\, N \tag{10}$$

(Un)vectorization

Let us unvectorize our starting (data) tensor, across dimension *Year*:

$$
\begin{array}{c}
\mathit{Year} \times (\mathit{Color} \times \mathit{Model}) \longleftarrow 1 \\
\boldsymbol{unvec}_{\mathit{Year}} \left(\begin{array}{lll|c}
 & & & \text{ALL} \\ \hline
1990 & \mathit{Blue} & \mathit{Chevy} & 87 \\
 & & \mathit{Ford} & 99 \\
 & \mathit{Green} & \mathit{Chevy} & 0 \\
 & & \mathit{Ford} & 64 \\
 & \mathit{Red} & \mathit{Chevy} & 5 \\
 & & \mathit{Ford} & 0 \\ \hline
1991 & \mathit{Blue} & \mathit{Chevy} & 0 \\
 & & \mathit{Ford} & 7 \\
 & \mathit{Green} & \mathit{Chevy} & 0 \\
 & & \mathit{Ford} & 0 \\
 & \mathit{Red} & \mathit{Chevy} & 0 \\
 & & \mathit{Ford} & 8
\end{array} \right)
\end{array}
=
\begin{array}{c}
\mathit{Color} \times \mathit{Model} \longleftarrow \mathit{Year} \\
\begin{array}{ll|cc}
 & & 1990 & 1991 \\ \hline
\mathit{Blue} & \mathit{Chevy} & 87 & 0 \\
 & \mathit{Ford} & 99 & 7 \\ \hline
\mathit{Green} & \mathit{Chevy} & 0 & 0 \\
 & \mathit{Ford} & 64 & 0 \\ \hline
\mathit{Red} & \mathit{Chevy} & 5 & 0 \\
 & \mathit{Ford} & 0 & 8
\end{array}
\end{array}
$$

There is room for further unvectorizing the outcome, this time across *Color* — next slide:

(De)vectorization

Further unvectorization:

$$Color \times Model \longleftarrow Year \qquad\qquad Model \longleftarrow Color \times Year$$

$$\mathbf{unvec}_{Color}\left(\begin{array}{ll|cc} & & 1990 & 1991 \\ \hline Blue & Chevy & 87 & 0 \\ & Ford & 99 & 7 \\ \hline Green & Chevy & 0 & 0 \\ & Ford & 64 & 0 \\ \hline Red & Chevy & 5 & 0 \\ & Ford & 0 & 8 \end{array}\right) =$$

	Blue		*Green*		*Red*	
	1990	1991	1990	1991	1990	1991
Chevy	87	0	0	0	5	0
Ford	99	7	64	0	0	8

and so on.

Generic cubes

It turns out **that** cubes can be calculated for any such two-dimensional versions of our original data tensor, for instance,

$$\textbf{cube } N : \ \mathit{Model} + 1 \longleftarrow (\mathit{Color} + 1) \times (\mathit{Year} + 1)$$
$$\textbf{cube } N = \tau_{\mathit{Model}} \cdot N \cdot (\tau_{\mathit{Color}} \otimes \tau_{\mathit{Year}})^{\circ}$$

where N stands for the second matrix of the previous slide, yielding

	Blue			*Green*			*Red*			ALL		
	1990	1991	ALL	1990	1991	ALL	1990	1991	ALL	1990	1991	ALL
Chevy	87	0	87	0	0	0	5	0	5	92	0	92
Ford	99	7	106	64	0	64	0	8	8	163	15	178
ALL	186	7	193	64	0	64	5	8	13	255	15	270

The 36 entries of the original cube have been rearranged in a 3*12 rectangular layout, as dictated by the **dimension** cardinalities.

The **cube** (LA) operator

Definition (Cube)

Let M be a matrix of type

$$\Pi_{j=1}^{n} B_j \xleftarrow{M} \Pi_{i=1}^{m} A_i \tag{11}$$

We define matrix **cube** M, the *cube of* M, as follows

$$\textbf{cube}\ M \;=\; (\bigotimes_{j=1}^{n} \tau_{B_j}) \cdot M \cdot (\bigotimes_{i=1}^{m} \tau_{A_i})^{\circ} \tag{12}$$

where $\bigotimes$ is finite Kronecker product.

So **cube** M has type $\Pi_{j=1}^{n}(B_j + 1) \longleftarrow \Pi_{i=1}^{m}(A_i + 1)$.

□

Properties of data cubing

Linearity:

$$\mathbf{cube}\ (M + N) \quad = \quad \mathbf{cube}\ M + \mathbf{cube}\ N \tag{13}$$

Proof: Immediate by bilinearity of matrix composition:

$$M \cdot (N + P) \quad = \quad M \cdot N + M \cdot P \tag{14}$$
$$(N + P) \cdot M \quad = \quad N \cdot M + P \cdot M \tag{15}$$

This can be taken advantage of not only in **incremental** data cube construction but also in **parallelizing** data cube generation.

Properties of data cubing

Updatability: by Khatri-Rao product linearity,

$$(M+N) \triangledown P = M \triangledown P + N \triangledown P$$
$$P \triangledown (M+N) = P \triangledown M + P \triangledown N$$

the **cube** operator commutes with the usual CRUDE operations, namely with record **updating**. For instance, suppose record

#	Model	Year	Color	Sale
5	*Ford*	1991	*Red*	8

cf

t_{Model}	1	2	3	4	5	6
Chevy	1	1	0	0	0	0
Ford	0	0	1	1	1	1

is updated to

#	Model	Year	Color	Sale
5	*Chevy*	1991	*Red*	8

cf

t'_{Model}	1	2	3	4	5	6
Chevy	1	1	0	0	1	0
Ford	0	0	1	1	0	1

Properties of data cubing

One just has to compute the “delta” projection,

$$\delta_{Model} = t'_{Model} - t_{Model} =$$

	1	2	3	4	5	6
Chevy	0	0	0	0	1	0
Ford	0	0	0	0	-1	0

then the “delta cube”,

$$d = (\tau_{Year} \otimes (\tau_{Color} \otimes \tau_{Model})) \cdot v'$$

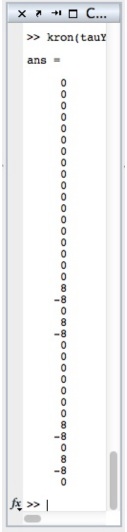
where

$$v' = (t_{Year} \triangledown (t_{Color} \triangledown \delta_{Model})) \cdot (t^{Sale})^{\circ}$$

and finally **add** the “delta cube” to the original cube:

$$c' = c + d.$$

— see the simulation aside (MATLab).



Properties of data cubing

Theorem (Cube commutes with vectorization)

Let $X \xleftarrow{M} Y \times C$ *and* $Y \times X \xleftarrow{\mathbf{vec}\,M} C$ *be its* Y*-vectorization. Then*

$$\mathbf{vec}\,(\mathbf{cube}\ M) = \mathbf{cube}\ (\mathbf{vec}\,M) \tag{16}$$

holds.

□

The proof (in the paper) relies on the type diagrams:

$$\begin{tikzcd}
Y \times X \arrow[d, "\tau_Y \otimes \tau_M"'] & & C \arrow[ll, "\mathbf{vec}_Y\ M"'] \arrow[r, phantom, "\cong"] & X \arrow[d, "\tau_X"'] & & Y \times C \arrow[ll, "M"'] \\
(Y+1) \times (X+1) & & C+1 \arrow[u, "\tau_C^\circ"'] \arrow[ll, shift right, "\mathbf{cube}\,(\mathbf{vec}_Y\ M)"'] \arrow[ll, shift left, "\mathbf{vec}_{Y+1}\,(\mathbf{cube}\ M)"] \arrow[r, phantom, "\cong"] & X+1 & & (Y+1) \times (C+1) \arrow[ll, "\mathbf{cube}\ M"'] \arrow[u, "(\tau_Y \otimes \tau_C)^\circ"']
\end{tikzcd}$$

Properties of data cubing

The following theorem shows that changing the dimensions of a data cube does not change its totals.

Theorem (Free theorem)

Let $B \xleftarrow{M} A$ *be cubed into* $B+1 \xleftarrow{\mathbf{cube}\ M} A+1$, *and* $r : C \to A$ *and* $s : D \to B$ *be arbitrary functions. Then*

$$\mathbf{cube}\ (s^\circ \cdot M \cdot r) \quad = \quad (s^\circ \oplus id) \cdot (\mathbf{cube}\ M) \cdot (r \oplus id) \tag{17}$$

holds, where $M \oplus N = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$ *is matrix* **direct sum**.

□

The proof given in the paper resorts to the **free theorem** of polymorphic operators popularized by Wadler (1989) under the heading *Theorems for free!*.

Cube universality — slicing

Slicing is a specialized filter for a particular value in a dimension.

Suppose that from our starting cube

$$c : 1 \rightarrow (Year + 1) \times ((Color + 1) \times (Model + 1))$$

one is only interested in the data concerning year 1991.

It suffices to regard data values as (categorial) **points**: given $p \in A$, constant function $\underline{p} : 1 \rightarrow A$ is said to be a *point* of A, for instance

$$\underline{1991} : 1 \rightarrow Year + 1$$

$$\underline{1991} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Cube universality — slicing

Example:

$$1 \xrightarrow{c} (\mathit{Year}+1)\times((\mathit{Color}+1)\times(\mathit{Model}+1)) \xrightarrow{\underline{1991}^\circ \otimes \mathit{id}} 1\times((\mathit{Color}+1)\times(\mathit{Model}+1)) \quad = \quad \begin{bmatrix}0\\7\\7\\0\\0\\0\\0\\8\\8\\0\\15\\15\end{bmatrix}$$

Cube universality — rolling-up

Gray et al. (1997) say that *going up the levels [of aggregated data] is called rolling-up*.

In this sense, a **roll-up** operation over dimensions A, B and C could be the following form of (increasing) summarization:

$$A \times (B \times C)$$

$$A \times B$$

$$A$$

$$1$$

How does this work over a data cube? We take the simpler case of two dimensions A, B as example.

Cube universality — rolling-up

The dimension powerset for A, B is captured by the corresponding matrix **injections** onto the cube target type $(A+1)\times(B+1)$:

$$\begin{tikzcd} & & (A+1)\times(B+1) & & \\ A\times B \arrow[urr, "\theta"] & & A \arrow[u, "\alpha"] & B \arrow[ul, "\beta"'] & 1 \arrow[ull, "\omega"'] \end{tikzcd}$$

where

$$\theta = i_1 \otimes i_1$$
$$\alpha = i_1 \triangledown i_2 \cdot !$$
$$\beta = i_1 \cdot ! \triangledown i_2$$
$$\omega = i_2 \triangledown i_2$$

NB: the injections i_1 and i_2 are such that $[i_1|i_2] = id$, where $[M|N]$ denotes the horizonal gluing of two matrices.

Cube universality — rolling-up

One can build compound injections, for instance

$$\rho : (A+1) \times (B+1) \leftarrow A \times B + (A+1)$$
$$\rho = [\theta| [\alpha|\omega]]$$

Then, for $M : C \to A \times B$:

$$\rho^\circ \cdot (\textbf{cube}\ M) = \left[\frac{M}{\left[\frac{fst \cdot M}{! \cdot M} \right]} \right] \cdot \tau_C^\circ$$

extracts from **cube** M the corresponding **roll-up**.

The next slides give a concrete example.

Cube universality — rolling-up

Let M be the (generalized) data cube

		1990	1991	ALL
	Chevy	87	0	87
Blue	*Ford*	99	7	106
	ALL	186	7	193
	Chevy	0	0	0
Green	*Ford*	64	0	64
	ALL	64	0	64
	Chevy	5	0	5
Red	*Ford*	0	8	8
	ALL	5	8	13
	Chevy	92	0	92
ALL	*Ford*	163	15	178
	ALL	255	15	270

Cube universality — rolling-up

Building the injection matrix $\rho = [\theta | \, [\alpha | \omega]]$ for types $\mathit{Color} \times \mathit{Model} + \mathit{Color} + 1 \rightarrow (\mathit{Color} + 1) \times (\mathit{Model} + 1)$ we get the following matrix (already transposed):

		Blue			Green			Red			ALL		
		Chevy	Ford	ALL	Chevy	Ford	ALL	Chevy	Ford	ALL	Chevy	Ford	ALL
Blue	Chevy	1	0	0	0	0	0	0	0	0	0	0	0
	Ford	0	1	0	0	0	0	0	0	0	0	0	0
Green	Chevy	0	0	0	1	0	0	0	0	0	0	0	0
	Ford	0	0	0	0	1	0	0	0	0	0	0	0
Red	Chevy	0	0	0	0	0	0	1	0	0	0	0	0
	Ford	0	0	0	0	0	0	0	1	0	0	0	0
	Blue	0	0	1	0	0	0	0	0	0	0	0	0
	Green	0	0	0	0	0	1	0	0	0	0	0	0
	Red	0	0	0	0	0	0	0	0	1	0	0	0
	ALL	0	0	0	0	0	0	0	0	0	0	0	1

Cube universality — rolling-up

Then

$$\rho^\circ \cdot \textbf{cube}\ M \;=\; \begin{array}{ll|ccc} & & 1990 & 1991 & \textsc{all} \\ \hline \multirow{2}{*}{\textit{Blue}} & \textit{Chevy} & 87 & 0 & 87 \\ & \textit{Ford} & 99 & 7 & 106 \\ \hline \multirow{2}{*}{\textit{Green}} & \textit{Chevy} & 0 & 0 & 0 \\ & \textit{Ford} & 64 & 0 & 64 \\ \hline \multirow{2}{*}{\textit{Red}} & \textit{Chevy} & 5 & 0 & 5 \\ & \textit{Ford} & 0 & 8 & 8 \\ \hline & \textit{Blue} & 186 & 7 & 193 \\ & \textit{Green} & 64 & 0 & 64 \\ & \textit{Red} & 5 & 8 & 13 \\ \hline & \textsc{all} & 255 & 15 & 270 \end{array}$$

Note how a roll-up is a particular "subset" of a cube.

Matrix ρ° performs the (quantitative) selection of such a subset.

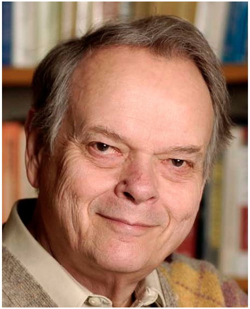
Summing up

Data science seems to be ignoring the role of **types** and **type parametricity** in software — one of the most significant advances in CS.

Nice theory called **parametric polymorphism** (John Reynolds, CMU).

So nice that you can derive **properties** of your operations **solely** by looking at their **types** 🙂

As Kurt Lewin (1890-1947) once write it: *"There is nothing more practical than a good theory"*.



J.C. Reynolds
(1935–2013)

Summing up

Abadir and Magnus (2005) stress on the need for a **standardized** notation for **linear algebra** in the field of econometrics and **statistics**.

This talk suggests such a notation should be **polymorphically typed**.

Since (Macedo and Oliveira, 2013) the author has invested in **typing** linear algebra in a way that makes it closer to modern **typed** languages.

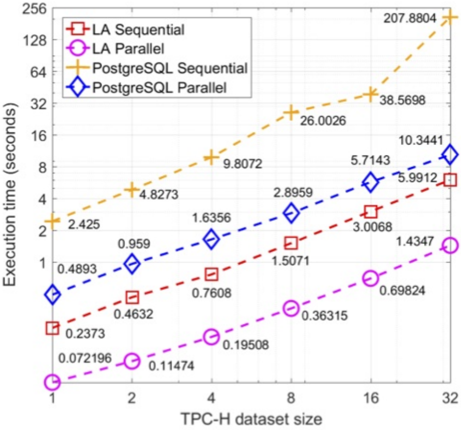
This extends previous efforts on applying LA to **OLAP** (Macedo and Oliveira, 2015)

(Still not convinced? Peek the next slide.)

Annex

(For those who care mostly about efficiency)

Aside: Plot taken from a recent MSc report on TPC-H benchmarking LA approach to analytical querying (on-going work).



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